AW-00

MATHEMATICAL FACTS AND FORMULÆ

BY

A. S. PERCIVAL, M.A.
Trinity College, Cambridge

BLACKIE & SON LIMITED
LONDON AND GLASGOW
1933

Com-03
AH-68.

MATHEMATICAL FACTS AND FORMULÆ

BY

A. S. PERCIVAL, M.A.
Trinity College, Cambridge

BLACKIE & SON LIMITED
LONDON AND GLASGOW
1933

BLACKIE & SON LIMITED
50 Old Bailey, London
17 Stanhope Street, Glasgow
BLACKIE & SON (INDIA) LIMITED
Warwick House, Fort Street, Bombay
BLACKIE & SON (CANADA) LIMITED
Toronto

Printed in Great Britain by Blackie & Son, Ltd., Glasgow

PREFACE

Much of the material in mathematical books is of no use to the practical worker, and what is of use is frequently forgotten. This little book is intended as a manual for laboratory workers, and is largely a collection of mathematical formulæ with easy proofs when thought necessary.

Simple methods based on common-sense principles are given in the case of Differential Equations with constant coefficients. I do not pretend to give rigid mathematical proofs in every case, but sufficient, I hope, to enable the reader to follow the reasoning and to remember the results.

I make little claim to originality. In two cases I owe much to correspondence with my old schoolmaster at Repton, the late Mr. J. H. Gurney: (1) the method of solving cubic and quartic equations with numerical coefficients by means of the usual logarithmic tables, and (2) the method of obtaining a general formula from a specific instance; this is quite simple, but I have never come across it in any book that I have seen.

I wish to thank the Publishers for the care they have taken in the production of the book.

A. S. PERCIVAL.

SHENLEY, WOKING,
15th August, 1933.

CONTENTS

MATHEMATICAL FACTS AND FORMULÆ

CHAPTER I

NOTANDA

Multiplication by Detached Coefficients.

If the coefficients are numerical this method reduces the written work. Each expression must be arranged in descending powers of some letter, say of x; when some powers of x are absent their places must be supplied by 0's.

E.g. $(3x^3 - 2x + 7)(2x^2 - 3)$

$$
\begin{array}{cccccc}
3 & 0 & -\ 2 & +\ 7 & & \\
2 & 0 & -\ 3 & & & \\
\hline
6 & 0 & -\ 4 & +\ 14 & & \\
 & 0 & 0 & 0 & & \\
 & & -\ 9 & 0 & +\ 6 & -\ 21 \\
\hline
6 & 0 & -\ 13 & +\ 14 & +\ 6 & -\ 21
\end{array}
$$

As the highest power in the product must be $3 + 2$ or 5, the product is $6x^5 - 13x^3 + 14x^2 + 6x - 21$.

Of course in the above case, as the second row consists entirely of 0's, it might be omitted provided that the last row is placed in its proper position, its first term being under the third term of the first row.

Division.

Division may be abridged in a somewhat similar way, which is found extremely useful. The coefficient of the first term of the divisor (i.e. the highest power of x in the divisor) must always be 1. For instance, suppose that we wish to divide

$$2x^4 + 11x^3 + 5x^2 - 26x + 8 \quad \text{by} \quad 2x^2 + 5x - 2,$$

we must regard the divisor as $x^2 + 2\frac{1}{2}x - 1$.

Write the coefficients of the dividend in a horizontal line, putting 0 where any term is wanting; draw a vertical line to the left of this series of coefficients and write in a vertical column the coefficients of the divisor with their signs changed, neglecting the first term of the divisor. From the foot of this vertical line draw another horizontal line; below this and under the first term of the dividend place its first coefficient, which is also the first coefficient of the quotient. Multiply each written term of the divisor by the first term of the quotient, and put down these products in the first of the oblique rows indicated by dotted lines. Add the two coefficients in the second column and their sum is the second coefficient of the quotient; put it down below the line in the second column. Now multiply each written term of the divisor by this second term, and write them down in the second oblique row. Add the three figures in the third column, and put down the sum below the line as the third coefficient of the quotient, and so on. Continue this until the last figure of an oblique row occurs under the last figure of the dividend.

E.g. $2x^4 + 11x^3 + 5x^2 - 26x + 8 \div 2x^2 + 5x - 2$. The divisor is to be $x^2 + 2\frac{1}{2}x - 1$.

$$\begin{array}{r|rrrrr}
 & 2 & +\,11 & +\,5 & -\,26 & +\,8 \\
-\,2\frac{1}{2} & & -\,5 & -\,15 & +\,20 & \\
+\,1 & & & +\,2 & +\,6 & -\,8 \\
\hline
 & 2 & +\,6 & -\,8 & 0 & 0
\end{array}$$

The dividend is of the 4th degree (m), the divisor of the 2nd degree (n), so the quotient will be of the 2nd degree ($m - n$). The quotient is then $2x^2 + 6x - 8$, but as the divisor is really twice as great as that used (being $2x^2 + 5x - 2$), the quotient will be $x^2 + 3x - 4$.

The fact that 2 zeroes represent the last two terms shows that the divisor must be a factor of the dividend; had they been significant, they would have represented the remainder.

On p. 86 it is required to put $D^4 + 2D^3 - 9D^2 - 2D + 8$ into factors. As there are two changes of sign there may be two positive *real* roots, and as the product of all the roots must be $8(-1)^4$, and their sum must be the coefficient of the second term with its sign changed (-2), we could try if -4 is a root; i.e. divide by $D + 4$. It is unnecessary to put -4 to the side; we merely use the value of the root. On finding this successful we next try if $D^2 - 1$ is a factor.

$$
\begin{array}{r|rrrrr}
 & 1 & +2 & -9 & -2 & +8 \\
 & & -4 & +8 & +4 & -8 \\
\hline
 & 1 & -2 & -1 & +2 & 0 \\
0 & & 0 & 0 & & \\
1 & & & +1 & -2 & \\
\hline
 & 1 & -2 & 0 & 0 &
\end{array}
$$

So the factors are $(D + 4)(D^2 - 1)(D - 2)$, the roots being $-4, \pm 1, +2$.

The method is also very convenient when tabulating or tracing functions; e.g. $y = x^3 - 5x^2 + 2x + 8$.

Whenever any value is given to x, the last term or remainder in the equation thus modified will give the value of y. For instance, when $x = 0$, $y = +8$; when $y = 0$, the curve crosses the axis of x, and that value of x is a root of the equation.

The adjoining table shows how work should be arranged.

$$\begin{array}{rl}
 & \qquad \underline{\underline{1 - 5 + 2 + 8}} \\
x = 1 & \qquad + 1 - 4 - 2 \\
 & \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
y = 6 & (1) \underline{- 4 - 2 + 6} \\
x = 2 & \qquad + 2 - 6 - 8 \\
 & \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
y = 0 & (1) \underline{- 3 - 4 \quad 0} \\
x = 3 & \qquad + 3 - 6 - 12 \\
 & \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
y = -4 & (1) \underline{- 2 - 4 - 4} \\
x = 4 & \qquad + 4 - 4 - 8 \\
 & \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
y = 0 & (1) \underline{- 1 - 2 \quad 0}
\end{array}$$

The original equation is denoted symbolically by $1 - 5 + 2 + 8$ doubly underlined.

First the original equation is divided by $x - 1$, with the symbolical quotient $(1) - 4 - 2$ with a remainder $+ 6$, which is the value of y when $x = 1$.

The original equation (doubly underlined) is then divided by $x - 2$ and the quotient is $(1) - 3 - 4$ or $x^2 - 3x - 4$ with no remainder; so when $x = 2$, $y = 0$.

When the original equation is divided by $x - 4$ or indeed by $x + 1$, it will be found that there is no remainder, so that the roots of the equation are $x = -1$, or 2 or 4, and at these points the curve crosses the axis of x, for then $y = 0$.

Compound Interest.

Dividends or interest on capital may be paid yearly, half-yearly, quarterly, or at the end of any other assigned term. The expression "rate per cent" always means the interest on £100 for one year.

Let r denote the interest paid on £1 at the end of one term, and let R denote the amount reached by £1 at the end of the term; then $R = 1 + r$. If P be the Principal or capital invested at compound interest, the amount M at the end of one term is PR, and at the end of n terms $M = PR^n$.

E.g. let $P =$ £250, let the rate per cent be 6, and let the dividends be paid half-yearly, i.e. £3 on every £100 half-yearly; so $r = \cdot 03$, and $R = 1\cdot 03$. What will be the amount at the end of 5 years if every dividend be added to the investment?

$$M = PR^{10} = 250(1\cdot 03)^{10} = 335\cdot 98$$
$$M = \pounds 335,\ 19s.\ 7d.$$

$$\begin{array}{lcr}
\log 1\cdot 03 & = & \cdot 01283722 \\
 & & 10 \\
\hline
 & & \cdot 1283722 \\
\log 250 & = & 2\cdot 39794 \\
\hline
\log 335\cdot 98 & = & 2\cdot 5263122
\end{array}$$

The following is a ready method of determining roughly how long it will take for the Principal to double itself at compound interest, when r per cent means the interest paid on £100 at the end of *each term*, whether it be paid yearly, half-yearly, or quarterly.

Let r per cent $= 2 + n$, then the number of terms required for the Principal to double itself is given very nearly by $\dfrac{70 + \frac{1}{3}n}{2 + n}$

So at 2% each quarter the time is $\dfrac{70}{2}$ or 35 quarters,

but at 8% paid yearly the time is $\dfrac{70 + \frac{1}{3}6}{2 + 6} = 9$ years,

at 1% each quarter the time is $\dfrac{70 - \frac{1}{3}}{2 - 1}$ or $69\frac{2}{3}$ quarters.

To indicate the degree of accuracy obtained by this method, the figures correct to five decimal places for some values of r per cent are given on p. 6.

r per cent	Terms	r per cent	Terms
1	69·66072	4	17·67364
2	35·00279	8	9·00647

Annuities.

The present value of an annuity of £A payable at the end of each of n successive terms is calculated in this way.

The present value of the first payment is $\dfrac{A}{R}$,

,, ,, ,, second ,, ,, $\dfrac{A}{R^2}$,

,, ,, ,, nth ,, ,, $\dfrac{A}{R^n}$.

$$P = \frac{A}{R}\left(1 + \frac{1}{R} + \frac{1}{R^2} + \dots \frac{1}{R^{n-1}}\right);$$

$$\therefore\ P = \frac{A}{R}\left(\frac{1 - \dfrac{1}{R^n}}{1 - \dfrac{1}{R}}\right) = \frac{A}{R-1}\left(1 - \frac{1}{R^n}\right).$$

E.g.: What is the present value of an annuity of £80, paid quarterly, for ten years, at a rate of 6 per cent?

Here $r = \dfrac{6}{400} = \cdot 015$, $R = 1\cdot 015$, and $A = £20$.

$$P = \frac{20}{\cdot 015}\left(1 - \frac{1}{(1\cdot 015)^{40}}\right) \qquad \begin{array}{r} \log 1\cdot 015 = \ \cdot 00646604 \\ 40 \\ \hline \cdot 2586416 \end{array}$$

$$\log (1\cdot 015)^{-40} = \log \cdot 55126 = \bar{1}\cdot 7413584$$

$$P = \frac{20}{\cdot 015}(1 - \cdot 55126) = \frac{4000}{3}(\cdot 44874)$$

$$= 598\cdot 32, \text{ say } £598,\ 6s.\ 5d.$$

The number of pence will not be quite accurate, as we have used only this table of five-figure logarithms for the evaluation of $\bar{1}{\cdot}7413584$, though the error will be slight. In this case the number of pence should be 4 and a tiny fraction of a penny. As the logarithm of R has to be multiplied by n, which may be very large, a list of logarithms to eight figures for the more usual values of R is given below.

It is clear that a bill of £598, 6*s*. would be justly paid off by quarterly payments of £20 for ten years if the rate of interest 6 per cent were accepted by both parties.

R	Log	R	Log
1·0025	·0010,8438	1·0325	·0138,9006
1·0050	·0021,6606	1·0350	·0149,4035
1·0075	·0032,4505	1·0375	·0159,8811
1·0100	·0043,2137	1·0400	·0170,3334
1·0125	·0053,9503	1·0425	·0180,7606
1·0150	·0064,6604	1·0450	·0191,1629
1·0175	·0075,3442	1·0475	·0201,5403
1·0200	·0086,0017	1·0500	·0211,8930
1·0225	·0096,6332	1·0525	·0222,2210
1·0250	·0107,2387	1·0550	·0232,5246
1·0275	·0117,8183	1·0575	·0242,8038
1·0300	·0128,3722	1·0600	·0253,0587

Approximations.

All instrumental readings are approximate, some to ·1 of a unit, some to ·001 of a unit, &c. If the reading be given as 11·2, it is assumed that reliance can only be placed on the first decimal place, so that it is best to take the given reading as $11{\cdot}2 \pm {\cdot}05$ ($\pm {\cdot}05$ being the limits of the variation); if the reading were given as 11·20, two decimal places should be regarded as accurate, and it should be taken as $11{\cdot}20 \pm {\cdot}005$. When it is necessary to combine such readings there is no difficulty if the readings are all of the same degree of accuracy.

Addition.

$$\begin{array}{l} 41{\cdot}3 \pm {\cdot}05 \\ \underline{11{\cdot}2 \pm {\cdot}05} \\ 52{\cdot}5 \pm {\cdot}1 \end{array}$$

Subtraction requires a little more thought. If the variations are taken as of the same sign, the result will be 30·1, but if of contrary sign the result will be either 30·2 or 30.

$$\begin{array}{l} 41{\cdot}3 \pm {\cdot}05 \\ \underline{11{\cdot}2 \pm {\cdot}05} \\ 30{\cdot}1 \pm {\cdot}1 \end{array}$$

It is seen that these results give the limits of the maximum or the minimum that might be attained, though, of course, it is far more probable that the true result will be in the neighbourhood of the mean value.

Multiplication.—Clearly

$$(11{\cdot}2 \pm {\cdot}05)\,(41{\cdot}3 \pm {\cdot}05)$$
$$= 462{\cdot}56 \pm {\cdot}05\,(41{\cdot}3 + 11{\cdot}2) + {\cdot}0025.$$

In this case, as only one decimal place is given in the data, the last term may be neglected, but it is well to retain two decimal places so as to obtain the right figure for the final decimal. The most correct solution will be $462{\cdot}56 \pm 2{\cdot}63$, giving $462{\cdot}6 \pm 2{\cdot}6$ as the only practical one that is fairly reliable to four significant figures.

If the data are of different accuracy, one can only rely upon the number of decimal places given in the least accurate measurement.

$$(A \pm h)\,(B \pm k) = AB \pm kA \pm hB \pm hk.$$

Let $h < k$, then since both h and k are fractional, $hk < k$, so the term hk must be neglected.

Suppose that we require to find an expression for

$$(11{\cdot}20 \pm {\cdot}005)\,(41{\cdot}3 \pm {\cdot}05) = 462{\cdot}56 \pm {\cdot}56 \pm {\cdot}2065 \pm {\cdot}00025,$$

or nearly $462{\cdot}56 \pm {\cdot}77$.

Now, although it is true that this statement gives us all that we need know, yet it is misleading, as it suggests that the result can be given to two places of decimals, whereas one can really only give it to one place of decimals. The solution should really be given as $462{\cdot}6 \pm {\cdot}8$, and to obtain this result no previous step need be taken to more than two decimal places.

Division.—This is troublesome; divide the maximum dividend by the minimum divisor and the minimum dividend by the maximum divisor, and so obtain both maximum and minimum quotient; one-half the difference of these quotients will be the variation, indicating the inaccuracy, which should have at least as many decimal places as the most accurate datum.

An example will explain the procedure; e.g. $\dfrac{41{\cdot}3 \pm {\cdot}05}{11{\cdot}20 \pm {\cdot}005}$.

$$\frac{41{\cdot}35}{11{\cdot}195} = 3{\cdot}69361 \qquad\qquad \frac{41{\cdot}25}{11{\cdot}205} = 3{\cdot}68139$$

Difference ·01222; halve it, say ·0061.
So quotient is $3{\cdot}6875 \pm {\cdot}0061$.

CHAPTER II

Algebraic Formulæ

Ratios.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$, obviously $\frac{a+c+e}{b+d+f} = k$.

Also $$\frac{\frac{a}{b} - 1}{\frac{a}{b} + 1} = \frac{\frac{c}{d} - 1}{\frac{c}{d} + 1} \quad \text{or} \quad \frac{a-b}{a+b} = \frac{c-d}{c+d}.$$

Progressions.

Arithmetical.—E.g.

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \ldots \quad a+(n-1)d.$$

To insert m means between a and b. Then b will be the $(m+2)$th term, and $d = \frac{b-a}{m+1}$.

If S be the sum of the series, and l is the nth or last term,

$$S = \frac{n}{2}(a+l) \quad \text{or} \quad \frac{n}{2}\left\{2a + (n-1)d\right\}.$$

Geometrical.—E.g. $a, ar, ar^2, ar^3, \ldots ar^{n-1}$.

To insert m geometric means between a and b. Then b will be the $(m+2)$th term, and $b = ar^{m+1}$, $\therefore r = \sqrt[m+1]{\frac{b}{a}}$.

$$S = a\frac{1-r^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

Hence if $r < 1$, S can differ from $\frac{a}{1-r}$ by as small a

quantity as we please by making n large enough. Hence if $r < 1$, the sum of an infinite number of terms is $\frac{a}{1-r}$.

Harmonical.—If a, b, c, d are in H.P.

$$\frac{a-b}{b-c} = \frac{a}{c}, \quad \frac{b-c}{c-d} = \frac{b}{d}, \quad \text{and so on.}$$

$\therefore\ c(a-b) = a(b-c)$; dividing by abc we have

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

$\therefore\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetical progression.

To insert m harmonic means between a and b. Insert m arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$, and the reciprocals of these will be the m harmonic means. The arithmetic means are:

$$\frac{1}{a} + \frac{1}{m+1}\left(\frac{1}{b} - \frac{1}{a}\right), \quad \frac{1}{a} + \frac{2}{m+1}\left(\frac{1}{b} - \frac{1}{a}\right),$$
$$\ldots, \frac{1}{a} + \frac{m}{m+1}\left(\frac{1}{b} - \frac{1}{a}\right).$$

The harmonic means are:

$$\frac{(m+1)ab}{mb+a}, \quad \frac{(m+1)ab}{(m-1)b+2a}, \quad \ldots \quad \frac{(m+1)ab}{b+ma}.$$

If A, G, H be the arithmetic, geometric, and harmonic means between any two quantities a and b,

$$A = \tfrac{1}{2}(a+b), \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}. \quad \therefore AH = G^2.$$

So the geometric mean between a and b is also the geometric mean between A and H.

The harmonic series, e.g. the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$, is *divergent*.

Continued Fractions.

To find convergents of $\frac{355}{113}$. Proceed as in finding H.C.F., but write down the successive quotients: 3, 7, 16. Below the first quotient write $\frac{1}{0}$, and let $\frac{0}{1}$ precede it.

	3	7	16	
$\frac{0}{1}$	$\frac{1}{0}$	$\frac{3}{1}$	$\frac{22}{7}$	$\frac{355}{113}$
$\frac{1}{0}$	$\frac{0}{1}$	$\frac{1}{3}$	$\frac{7}{22}$	$\frac{113}{355}$

Multiply the numerator by the quotient above it, add the preceding numerator and write the resulting number under the next quotient, and so on. The denominator is found by a precisely similar rule.

If $\frac{1}{0}$ and $\frac{0}{1}$ be interchanged, a proper fraction, the reciprocal of the previous improper fraction, is obtained.

Note that if $\frac{a}{b}$ be the last convergent considered, and $\frac{p}{q}$ be the penultimate convergent, $aq - bp = \pm 1$. This property is important in the solution of Indeterminate Equations.

Indeterminate Equations of the first degree.

The equation $ax \pm by = c$ can be satisfied by an infinite number of values of x and y; but if a and b are prime to each other and if a, b, and c are integers, there will only be a limited number of positive integral values of x and y that satisfy the equation $ax + by = c$. This is the procedure.

(1) $ax + by = c$. If $\frac{a}{b}$ be the last convergent, and $\frac{p}{q}$ be the penultimate convergent, $aq - bp = \pm 1$.

(i) When $aq - bp = 1$.

Then $ax + by = c(aq - bp)$, so

$$a(x - cq) = -b(y + cp), \text{ or } x - cq = -\frac{b}{a}(y + cp).$$

Both sides of the equation are integral, so a must be a factor of $y + cp$. Let $y + cp = am$ where m is any integer. Then

$$y = am - cp \quad \text{and} \quad x = cq - bm.$$

But if only positive integral values of x and y are admissible, bm must not be greater than cq, and am must be greater than cp. An example will make this clear.

In how many ways can 25*s*. 6*d*. be paid in half-crowns and florins?

$$2\tfrac{1}{2}x + 2y = 25\tfrac{1}{2}, \quad \text{or} \quad 5x + 4y = 51.$$

Here $\dfrac{a}{b} = \dfrac{5}{4}$ and $\dfrac{p}{q} = \dfrac{1}{1}$. $\quad \therefore\ aq - bp = 1.$

$x = cq - bm = 51 - 4m.$
$y = -cp + am = -51 + 5m.$
To keep x positive $m < 13$.
To make y positive $m > 10$.

m	11	12
x	7	3
y	4	9

This is the simplest way of determining the number of positive solutions.

(ii) When $aq - bp = -1$, $x = -cq + bm$, and $y = cp - am$.

If only positive integral solutions are required,

$$bm > cq \quad \text{and} \quad am < cp;$$

but there may be no positive solutions.

(2) $ax - by = c$. An infinite number of positive integral values can be assigned to x and y by giving m values that make both x and y positive. It is more often required to find the smallest values of x and y that will satisfy the equation.

(i) When $aq - bp = 1$:

$$x = cq + bm. \qquad y = cp + am.$$

To find the lowest permissible positive values of x and y, m must be given the highest negative value that will leave x and y positive; i.e. $-bm$ must not be greater numerically than cq, nor $-am$ greater than cp.

E.g. $5x - 4y = 51$; $\therefore\ x = 51 + 4m$ and $y = 51 + 5m$.

Here the value of -10 for m will give the lowest positive values of both x and y, $x = 11$ and $y = 1$.

The general solution will be

$$\begin{aligned} x &= 11 + 4m, \\ y &= \ \ 1 + 5m. \end{aligned}$$

(ii) When $aq - bp = -1$:

$$x = -cq + bm. \qquad y = -cp + am. \qquad \text{E.g. } 7x - 4y = 51.$$

Here $\frac{a}{b} = \frac{7}{4}$ and $\frac{p}{q} = \frac{2}{1}$; $aq - bp = -1$;

$$\therefore\ x = -51 + 4m \text{ and } y = -102 + 7m.$$

So $m = 15$ will give the lowest positive values of x and y; $x = 9$ and $y = 3$. The general solution will be

$$x = 9 + 4m,$$
$$y = 3 + 7m.$$

Simultaneous Equations.

(1) *Homogeneous* equations in x and y. The ratio $\frac{x}{y}$ or $\frac{y}{x}$ can always be found: e.g. $24x^2 + y^2 = 2x(5y + 4x)$.

Let $y = rx$, and divide by x^2. Then $r^2 - 10r + 16 = 0$.

$$\therefore\ r \text{ or } \frac{y}{x} = 8 \text{ or } 2,\ y = 8x \text{ or } 2x.$$

(2) (i) When the sum or difference of powers of the unknowns is symmetrically given, the following devices are often useful.

$x + y = s, \quad xy = p.$	$x - y = d, \quad xy = p.$
$x^2 + y^2 = s^2 - 2p.$	$x^2 + y^2 = d^2 + 2p.$
$x^3 + y^3 = s^3 - 3ps.$	$x^3 - y^3 = d^3 + 3pd.$
$x^4 + y^4 = s^4 - 4ps^2 + 2p^2.$	$x^4 + y^4 = d^4 + 4pd^2 + 2p^2.$
$x^5 + y^5 = s^5 - 5ps^3 + 5p^2s.$	$x^5 - y^5 = d^5 + 5pd^3 + 5p^2d.$

(ii) Or let $u = \frac{1}{2}(x + y)$, and $v = \frac{1}{2}(x - y)$.

$x + y = 2u.$	$x - y = 2v.$
$x^2 + y^2 = 2u^2 + 2v^2.$	$x^2 - y^2 = 4uv.$
$x^3 + y^3 = 2u^3 + 6uv^2.$	$x^3 - y^3 = 6u^2v + 2v^3.$
$x^4 + y^4 = 2u^4 + 12u^2v^2 + 2v^4.$	$x^4 - y^4 = 8u^3v + 8uv^3.$
$x^5 + y^5 = 2u^5 + 20u^3v^2 + 10uv^4.$	$x^5 - y^5 = 10u^4v + 20u^2v^3 + 2v^5.$

By either of these methods it can be easily seen that the

solution of $x^3 + y^3 = 133$, and $x + y = 7$, is $x = 5$, $y = 2$; or $x = 2$, $y = 5$.

(3) When the inverse of all the unknowns is given, let u and v denote the reciprocals and clear of fractions;

$$\text{e.g. } \frac{1}{x} + \frac{1}{2y} = \frac{2}{3}, \qquad \frac{1}{3x} + \frac{2}{y} = \frac{5}{6}.$$

$$\therefore\ 6u + 3v = 4, \text{ and } 2u + 12v = 5,\ u = \frac{1}{2},\ v = \frac{1}{3};\ x = 2,\ y = 3.$$

(4) When two equations of the first order are given in x and y, e.g. (1) $ax + by = k$, and (2) $a'x + b'y = k'$, on multiplying throughout (1) by b', and (2) by $-b$, and then adding these equations, the term involving y vanishes, and we are left with $(ab' - a'b)x = kb' - k'b$. We may eliminate x by using the same procedure with the coefficients of y, and we then obtain $(a'b - ab')y = ka' - k'a$. It is more convenient to change the signs, and write $(ab' - a'b)y = k'a - ka'$, for then the solution can be written in a symmetrical determinant form.

$$x\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = \begin{vmatrix} k & b \\ k' & b' \end{vmatrix} \text{ and } y\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = \begin{vmatrix} a & k \\ a' & k' \end{vmatrix},$$

or $x(ab' - a'b) = kb' - k'b$ and $y(ab' - a'b) = ak' - a'k$.

Note that in the final determinants k and k' replace the coefficients of the unknown (x or y) required.

We know that when two simultaneous equations are given, the coefficients of one of the unknowns can be made to vanish; in a similar way when three simultaneous equations are given, two of the unknowns may be made to vanish.

$$\begin{aligned} \text{E.g. (1) } & A_1x + B_1y + C_1z + D_1v = k_1, \\ \text{(2) } & A_2x + B_2y + C_2z + D_2v = k_2, \\ \text{(3) } & A_3x + B_3y + C_3z + D_3v = k_3. \end{aligned}$$

To eliminate z and v, multiply each equation by the determinant formed by the coefficients of z and v in the other equations taken in cyclical order, and add the equa-

tions together; the coefficients of z and v will vanish, and we obtain only an equation in x and y.

If we multiply

$$(1) \text{ by } \begin{vmatrix} C_2 & D_2 \\ C_3 & D_3 \end{vmatrix}, \quad (2) \text{ by } \begin{vmatrix} C_3 & D_3 \\ C_1 & D_1 \end{vmatrix}, \quad (3) \text{ by } \begin{vmatrix} C_1 & D_1 \\ C_2 & D_2 \end{vmatrix},$$

and then add the equations, the coefficient of z will be

$$C_1(C_2D_3 - C_3D_2) + C_2(C_3D_1 - C_1D_3) \\ + C_3(C_1D_2 - C_2D_1) = 0,$$

and that of v will be

$$D_1(C_2D_3 - C_3D_2) + D_2(C_3D_1 - C_1D_3) \\ + D_3(C_1D_2 - C_2D_1) = 0.$$

An example will make the procedure clear:

$$\begin{array}{lrl} (1) & 3x - y - 2z + v &= 0; \\ (2) & -2x + 3y + z - v &= 9; \\ (3) & x - y + z - 2v &= 4; \\ (4) & 4x + 2y - 3z + 3v &= 4. \end{array}$$

To eliminate z and v we take equations (1), (2), and (3) first.

Multiply (1) by

$$\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} \text{ or } -1, \; (2) \text{ by } \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \text{ or } -3, \; (3) \text{ by } \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} \text{ or } 1.$$

As the z and v terms will finally vanish, these need not be multiplied.

$$\begin{array}{lrl} & -3x + y \ldots &= 0 \\ & 6x - 9y \ldots &= -27 \\ & x - y \ldots &= 4 \\ \hline (A) & 4x - 9y &= -23 \end{array}$$

To evaluate x and y another triplet must be taken from the original equations including (4), suppose (2), (3), and (4).

Multiply (2) by $\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix}$ or -3, (3) by $\begin{vmatrix} -3 & 3 \\ 1 & -1 \end{vmatrix}$ or 0,

(4) by $\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix}$ or -1.

We have then only to deal with (2) and (4), and we may neglect the z and v terms.

$$\begin{array}{lrl} & 6x - \ \ 9y \ldots = & -27 \\ & -4x - \ \ 2y \ldots = & -\ \ 4 \\ \hline \text{(B)} & 2x - 11y \ldots = & -31 \end{array} \qquad \begin{array}{lrl} \text{(A)} & 4x - \ \ 9y = & -23 \\ \text{from (B)} & -4x + 22y = & \ \ 62 \\ \hline & 13y = & \ \ 39 \end{array}$$

$$x = 1,\ y = 3,\ z = -2,\ v = -4.$$

When n simultaneous equations are given with n unknowns, two of the unknowns can be eliminated by this procedure, and we are left with $n - 2$ equations with $n - 2$ unknowns.

Quadratic Equations.

$$Ax^2 + Bx + C = 0.$$

$$2Ax + B = \pm\sqrt{B^2 - 4AC}, \text{ or } x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A};$$

$$\text{if } A = 1,\ x = -\frac{B}{2} \pm \sqrt{\left(\frac{B}{2}\right)^2 - C}.$$

Cubic Equations (Numerical Coefficients).

$$Ax^3 + Bx^2 + Cx + D = 0.$$

Eliminate the second term by substituting $\dfrac{y - B}{3A}$ for x.

Then $$y^3 + py + q = 0;$$
if $p = 9AC - 3B^2$, and $q = 2B^3 - 9A(BC - 3AD)$.

(If $A = 1$, and $\frac{1}{3}B$ be an integer n; let $x = y - n$, $p = C - 3n^2$, $q = 2n^3 - nC + D$.)

Let $\dfrac{\frac{1}{2}q}{\frac{p}{3}\sqrt{\frac{p}{3}}} = k$, k being regarded as positive in every case.

(A) When p is positive, $y^3 + py \pm q = 0$.

Put $k = \cot\psi$. Then $y_1 = \mp\sqrt{\frac{p}{3}}\left\{\cot^{\frac{1}{3}}\frac{\psi}{2} - \tan^{\frac{1}{3}}\frac{\psi}{2}\right\}$,

$$y_2 \text{ and } y_3 = -\frac{y_1}{2} \pm \sqrt{-p - 3\left(\frac{y_1}{2}\right)^2}.$$

(B) When p is negative, $y^3 - py \pm q = 0$.

(1) If $k > 1$, put $k = \operatorname{cosec}\psi$,

$$y_1 = \mp \sqrt{\frac{p}{3}}\left\{\cot^{\frac{1}{3}}\frac{\psi}{2} + \tan^{\frac{1}{3}}\frac{\psi}{2}\right\},$$

$$y_2 \text{ and } y_3 = -\frac{y_1}{2} \pm \sqrt{p - 3\left(\frac{y_1}{2}\right)^2}.$$

2) If $k = 1$,

$$y = \mp 2\sqrt{\frac{p}{3}},\ y_2 = y_3 = \pm\sqrt{\frac{p}{3}}.$$

(3) If $k < 1$, let $k = \cos 3\phi$,

$${}_1 = \mp 2\sqrt{\frac{p}{3}}\cos\phi,\ y_2 \text{ and } y_3 = \mp 2\sqrt{\frac{p}{3}}\cos(\phi \pm 120°).$$

The unreal roots may be very important in differential equations.

Quartic Equations (Numerical Coefficients).

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$
$$= F(y) = y^4 + Qy^2 + Ry + S = 0,$$

if
$$x = \frac{y - B}{4A},$$

when $Q = 16AC - 6B^2$, $R = 8(B^3 - 4ABC + 8A^2D)$, and $S = 16A(B^2C - 4ABD + 16A^2E) - 3B^4$.

If $A = 1$, and $\frac{1}{4}B$ be an integer n, let $x = y - n$, then

$$Q = C - 6n^2,\quad R = D - 2n(C - 4n^2),$$
$$S = E - nD + n^2(C - 3n^2).$$

The Auxiliary Cubic is $z^6 + 2Qz^4 + (Q^2 - 4S)z^2 - R^2 = 0$. This must at least have one positive root; let it be a^2.

Then $$(z^2 - a^2)(z^4 + pz^2 + q) = 0,$$

and $$F(y) = \left(y^2 + ay + \frac{a^2 + p - 2\sqrt{q}}{4}\right)\left(y^2 - ay + \frac{a^2 + p + 2\sqrt{q}}{4}\right) = 0,$$

when $\sqrt{q}$ takes the sign of R, and p retains its sign in the Auxiliary Cubic.

E.g. $$y^4 - 694y^2 + 2856y + 51597 = 0.$$

$$\text{Aux. cubic } (z^2 - 1156)(z^4 - 232z^2 + 7056) = 0.$$

$$\therefore (y^2 + 34y + 189)(y^2 - 34y + 273) = 0.$$

$$\therefore y = 21, 13, -7, \text{ or } -27.$$

Note (1): If cubic has 3 real, positive roots, quartic has 4 real roots.

Note (2): If cubic has 2 unreal roots, quartic has 2 real and 2 unreal roots.

Note (3): If cubic has 2 negative roots, quartic has either 2 or 4 unreal roots.

Approximations (Barlow's Method).

$$x^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + \ldots = k.$$

Let r be an approximate root so that

$$r^n + Br^{n-1} + Cr^{n-2} + Dr^{n-3} + \ldots = v.$$

$$x \approx r + \frac{2r(k-v)}{(n-1)(k \text{ or } v) + (n+1)r^n + (n-1)Br^{n-1} + (n-3)Cr^{n-2} + \ldots}.$$

In the denominator use k if $r > 1$ in absolute value irrespective of sign; use v if $r < 1$. The approximation will practically double the number of figures after every operation.

In every case in which the first two or three figures

of a root have been found, the remaining figures to any desired number of decimal places can be accurately determined by Horner's method.

Reciprocal Equations.

I. Coefficients the same whether read in order backwards or forwards.

II. Coefficients the same, but reversed in sign.

I. (1) *Of even degree,*

e.g. $$x^4 - 9{\cdot}45x^3 + 24{\cdot}1x^2 - 9{\cdot}45x + 1 = 0.$$

Divide by x^2,

$$\left(x^2 + \frac{1}{x^2}\right) - 9{\cdot}45\left(x + \frac{1}{x}\right) + 24{\cdot}1 = 0.$$

Let $x + \dfrac{1}{x} = y$, then $y^2 - 2 = x^2 + \dfrac{1}{x^2}$;

$$\therefore\ y^2 - 9{\cdot}45y + 22{\cdot}1 = 0.$$

Hence $y = 5{\cdot}2$ or $4{\cdot}25$ and $x = 5, \frac{1}{5}, 4$, and $\frac{1}{4}$.

I. (2) *Of odd degree.*

Divide by $x + 1$, so one root is -1, and the remaining equation is in form I (1).

II. (1) *Of even degree.*

All equations of this class lack the middle term.

Divide by $x^2 - 1$, and an equation of class I (1) is obtained.

$$\begin{array}{r|l} +1 & x^6 - 9{\cdot}45x^5 + 23{\cdot}1x^4 \qquad\qquad - 23{\cdot}1x^2 + 9{\cdot}45x - 1 = 0 \\ & \qquad\qquad\qquad 1 \quad -9{\cdot}45 \quad +24{\cdot}1 \quad -9{\cdot}45 \quad +1 \\ \hline & x^4 - 9{\cdot}45x^3 + 24{\cdot}1x^2 - 9{\cdot}45x + 1 = 0 \end{array}$$

$$x = \pm 1, 5, \tfrac{1}{5}, 4, \tfrac{1}{4} \text{ as before.}$$

II. (2) *Of odd degree.*

Divide by $x - 1$, and an equation of class I (1) is obtained.

$$\begin{array}{r|l} +1 & x^5 - 10{\cdot}45x^4 + 33{\cdot}55x^3 - 33{\cdot}55x + 10{\cdot}45x - 1 = 0 \\ & \quad + 1 \quad - 9{\cdot}45 \quad + 24{\cdot}1 \quad - 9{\cdot}45 \quad + 1 \\ \hline & x^4 - 9{\cdot}45x^3 + 24{\cdot}1x^2 - 9{\cdot}45x + 1 = 0 \end{array}$$

$$x = 1, 5, \tfrac{1}{5}, 4, \tfrac{1}{4}.$$

Quasi-Reciprocal Equations in which the signs of corresponding coefficients are irregular. A similar method may be successful.

$$x^6 - 4x^5 - 11x^4 + 40x^3 + 11x^2 - 4x - 1 = 0.$$

Divide by x^3,

$$x^3 - \frac{1}{x^3} - 4\left(x^2 + \frac{1}{x^2}\right) - 11\left(x - \frac{1}{x}\right) + 40 = 0.$$

Let $y = x - \dfrac{1}{x}$,

then $\quad y^2 + 2 = x^2 + \dfrac{1}{x^2}$, and $y^3 + 3y = x^3 - \dfrac{1}{x^3}$;

$$y^3 + 3y - 4(y^2 + 2) - 11y + 40 = y^3 - 4y^2 - 8y + 32 = 0.$$

One root of this is clearly 4,

$$\begin{array}{rrrr} & 4 & 0 & -32 \\ \hline y^2 & 0 & -8 & = 0 \end{array}$$

$$y = 4, \text{ or } \pm 2\sqrt{2}.$$

Then $\quad x^2 - 4x - 1 = 0$ or $x = 2 \pm \sqrt{5}$,

and $\quad x^2 \mp 2\sqrt{2}x - 1 = 0$ or $x = \pm\sqrt{2} \pm \sqrt{3}$.

Permutations and Combinations.

The number of ways in which r things can be taken from n things and arranged in a different order is called the number of permutations of n things r at a time, and is denoted by ${}_nP_r$.

$${}_nP_r = n(n-1)(n-2)\ldots(n-r+1)$$
$${}_nP_n = n(n-1)(n-2)\ldots 4\,.\,3\,.\,2\,.\,1 = \underline{|n}.$$

If the n things are not all different, suppose p of them to be a's, q of them to be b's, r of them to be c's, then the number of permutations will be $\dfrac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}$ if they are all taken together.

The number of ways in which r things can be taken from n different things without regard to their order is called the number of combinations of n things taken r at a time, and is denoted by ${}_nC_r$.

$$ {}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}. $$

Note that $\underline{|n} = n\,\underline{|n-1}$ in every case where n is a positive integer.

If we assume that the result is still true for $n = 1$, then

$$ \underline{|1} = 1\,\underline{|0}; \quad \therefore \underline{|0} = 1. $$

The number of combinations of n different things taken r together is equal to the number of the combinations taken $n - r$ together.

$$ \text{For } {}_nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \text{ and } {}_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}. $$

Powers of Binomials. $(a + b)^n$.

Let $a > b$, and let $\dfrac{b}{a} = x$. Then $(a + b)^n = a^n(1 + x)^n$. This is the standard form, for $(1 + x)^n$ is convergent when $x < 1$ numerically.

$$ a^n(1+x)^n = a^n\Big(1 + nx + \frac{n\,.\,\overline{n-1}}{\underline{|2}}x^2 + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{\underline{|3}}x^3 + \ldots x^n\Big), $$

or if n be a positive integer the series may be written

$$ a^n(1 + {}_nC_1x + {}_nC_2x^2 + {}_nC_3\,x^3 + \ldots x^n). $$

Note that the coefficient of x^r, in the $(r+1)$th term, is

$$\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,\ldots\,\overline{n-r+1}}{\underline{|r}},$$

and is the coefficient of x^{r-1} multiplied by $\left(\dfrac{n+1}{r}-1\right)x$.

If x be negative, and the index be a negative integer, the expression is even simpler, e.g.

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+1)x^r + \ldots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \frac{\overline{r+1}\,.\,\overline{r+2}}{\underline{|2}}x^r + \ldots$$

$$(1-x)^{-4} = 1 + 4x + 10x^2 + 20x^3 + \ldots + \frac{\overline{r+1}\,.\,\overline{r+2}\,.\,\overline{r+3}}{\underline{|3}}x^r + \ldots$$

If x be positive, the signs of the odd powers of x must be changed. Such expressions are often useful when x is very small, say ·005 or so.

The square roots of $1+x$ and $1-x$ or their reciprocals are often required, so they are given below.

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5 - \ldots$$

$$(1-x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 - \tfrac{7}{256}x^5 - \ldots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \tfrac{35}{128}x^4 - \tfrac{63}{256}x^5 + \ldots$$

$$(1-x)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{5}{16}x^3 + \tfrac{35}{128}x^4 + \tfrac{63}{256}x^5 + \ldots$$

Note that whenever r is greater than $n+1$, $\dfrac{n+1}{r}-1$ must be negative; hence if x be *negative*, all succeeding terms after this value of r are of the *same* sign as the rth; but if x be *positive*, all succeeding terms after the rth will be *alternately positive and negative.*

Exponential Series.

Use the Greek letter ϵ when the series is indicated, and the English e when the number 2·71828.. is denoted.

$$\epsilon^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \dots$$

This series is convergent for all values of x.

$$\underset{n\to\infty}{L}\left(1+\frac{1}{n}\right)^n = e = 1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\dots = 2{\cdot}7182818.$$

Logarithmic Series to base e. Write $\lg n$ for $\log_e n$. If $y^2 < 1$,

$$\lg(1+y) = \quad y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \dots$$

$$\lg(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \frac{y^5}{5} - \dots$$

$$\lg\frac{1+y}{1-y} = 2\left\{y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \dots\right\}$$

Let $y = \dfrac{m-n}{m+n}$; $\therefore\ \dfrac{1+y}{1-y} = \dfrac{m}{n}$,

$$\lg\frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \tfrac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \tfrac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \dots\right\}$$

$$\log n = \log e(\lg n) \text{ or } M \lg n.$$

$$M \text{ or } \log e = {\cdot}434294\dots \quad \frac{1}{M} = 2{\cdot}302585.. = \lg 10.$$

(See Table of Conversion Factors, p. 112).

For a method of deducing all the above series see p. 54 (Maclaurin's Theorem).

CHAPTER III

ANALYTICAL GEOMETRY (RECTANGULAR CO-ORDINATES)

The locus of every general equation of the second degree, e.g. $ax^2 + bxy + cy^2 + dx + ey + f = 0$, is either a conic section or two straight lines real or unreal.

In every equation of any order:

(1) If there be no absolute term (f), the locus or curve passes through the origin; in that case the tangent at the origin will be given by equating to zero the term or terms of lowest degree. E.g. $y^2 - 4ax = 0$; $4ax = 0$, so the axis of y is a tangent at the origin.

(2) *Symmetry.*—If no odd powers of x occur, the curve is symmetrical with respect to the axis of y; similarly for symmetry about the axis of x.

E.g. in $y^2 = 4ax$ there is symmetry about the axis of x. If on changing the signs of x and y the equation remains unchanged, there is symmetry in opposite quadrants; e.g. in the hyperbola $xy = f$.

In $y = x^3 - x$, the tangent at the origin is $y = -x$, there is symmetry in opposite quadrants, and y is a maximum where $x = -\dfrac{1}{\sqrt{3}}$, and a minimum where $x = \dfrac{1}{\sqrt{3}}$.

In equations of the second degree, if the locus has a centre its co-ordinates are

$$x_c = \frac{2cd - be}{b^2 - 4ac}, \qquad y_c = \frac{2ae - bd}{b^2 - 4ac}.$$

If $b^2 = 4ac$ (no centre), parabola or two parallel lines,
$b^2 > 4ac$, hyperbola or two intersecting lines,
$b^2 < 4ac$, ellipse or circle.

Simplification of the general equation of the second degree.

A. *When there is a centre,* $b^2 \neq 4ac$.

(1) Remove the x and y terms of the first degree; i.e. transfer the origin to the centre. This can always be done by writing $X + x_c$ for x, and $Y + y_c$ for y.

E.g. $$9x^2 - 16y^2 + 18x = 135.$$

Here $$x_c = \frac{-32(18)}{36(16)} = -1, \quad y_c = 0,$$

$$9(X-1)^2 - 16Y^2 + 18(X-1) = 135,$$

or $$9X^2 - 16Y^2 = 144, \text{ or } \frac{X^2}{16} - \frac{Y^2}{9} = 1,$$

a hyperbola. When $b^2 - 4ac$ is positive, the locus may be two intersecting lines; in such a case the centre is represented by the point of intersection, and on transferring the origin to this point the absolute term vanishes.

E.g. $$8x^2 - 6xy + y^2 - 6x + 2y + 1 = 0.$$

Here $$x_c = 0, \quad y_c = \frac{32 - 36}{36 - 32} = -1,$$

$$8X^2 - 6X(Y-1) + (Y-1)^2 - 6X + 2Y - 2 + 1 = 0,$$

or $$8X^2 - 6XY + Y^2 = 0.$$

Solve the quadratic $8x^2 - 6x + 1 = 0$, $x = \frac{1}{2}$ or $\frac{1}{4}$, so $X = \frac{1}{2}Y$ or $\frac{1}{4}Y$; $\therefore$ $Y = 2X$ and $Y = 4X$ are the two intersecting lines at the new origin.

(2) If after the above procedure an absolute term persist as well as the xy term, the xy term must be removed by rotating the locus round its centre through the angle θ. This is effected by operating upon the equation

$$ax^2 + bxy + cy^2 = f,$$

by the versor $\epsilon^{i\theta}$ or $\cos\theta + i\sin\theta$ (p. 32).

It must be remembered that $\sqrt{-1}$ or i rotates x and y counter-clockwise through $\frac{1}{2}\pi$, so $ix = y$ and $iy = -x$. The angle θ can always be found from the expression $\tan 2\theta = \dfrac{b}{c-a}$.

Let the equation be

$$10{\cdot}75\, x^2 + 3{\cdot}5\sqrt{3}\, xy + 14{\cdot}25\, y^2 = 144,$$

$$\tan 2\theta = \frac{3{\cdot}5\sqrt{3}}{14{\cdot}25 - 10{\cdot}75} = \sqrt{3}.$$

$2\theta = n\pi + \frac{1}{3}\pi$, say $\theta = 30°$, $120°$, $210°$, or $300°$; let $30°$ be taken. Now θ being positive denotes a $30°$ counter-clockwise rotation of the locus (in this case an ellipse) round its centre. All the terms of the equation that involve x or y must be multiplied by $\epsilon^{i\theta}$ or $\cos\theta + i\sin\theta$; had θ been negative the versor would have been $\epsilon^{-i\theta}$ or $\cos\theta - i\sin\theta$.

$$10{\cdot}75\left(\frac{X\sqrt{3}}{2} + \frac{Y}{2}\right)^2 + 3{\cdot}5\sqrt{3}\left(\frac{X\sqrt{3}}{2} + \frac{Y}{2}\right)\left(\frac{Y\sqrt{3}}{2} - \frac{X}{2}\right)$$
$$+\ 14{\cdot}25\left(\frac{Y\sqrt{3}}{2} - \frac{X}{2}\right)^2 = 144,$$

or

$$\tfrac{3}{4}X^2(10{\cdot}75 - 3{\cdot}5 + 4{\cdot}75) + \tfrac{1}{2}\sqrt{3}XY(10{\cdot}75 + 3{\cdot}5 - 14{\cdot}25)$$
$$+\ \tfrac{1}{4}Y^2(10{\cdot}75 + 10{\cdot}5 + 42{\cdot}75) = 144,$$

$$9X^2 + 16Y^2 = 144, \text{ or } \frac{X^2}{16} + \frac{Y^2}{9} = 1.$$

Here we have rotated the ellipse positively, with reference to the axes, but the result may be regarded as a rotation of the *axes negatively*.

To convert X and Y into corresponding values of x and y, we must rotate the *axes* positively or use $\epsilon^{i\theta}$ again. Take $X = 4$, $Y = 0$.

$$x = X\cos\theta + Y\sin\theta = 2\sqrt{3}.$$
$$y = Y\cos\theta - X\sin\theta = -2.$$

When we are dealing with a *point* X, Y, and finding the value of x and of y in terms of X and Y by using the versor $\epsilon^{i\theta}$, we are really rotating the axes positively. Confusion will arise unless we remember on what we are operating—whether individual points or the locus of an equation.

The standard equation for the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,

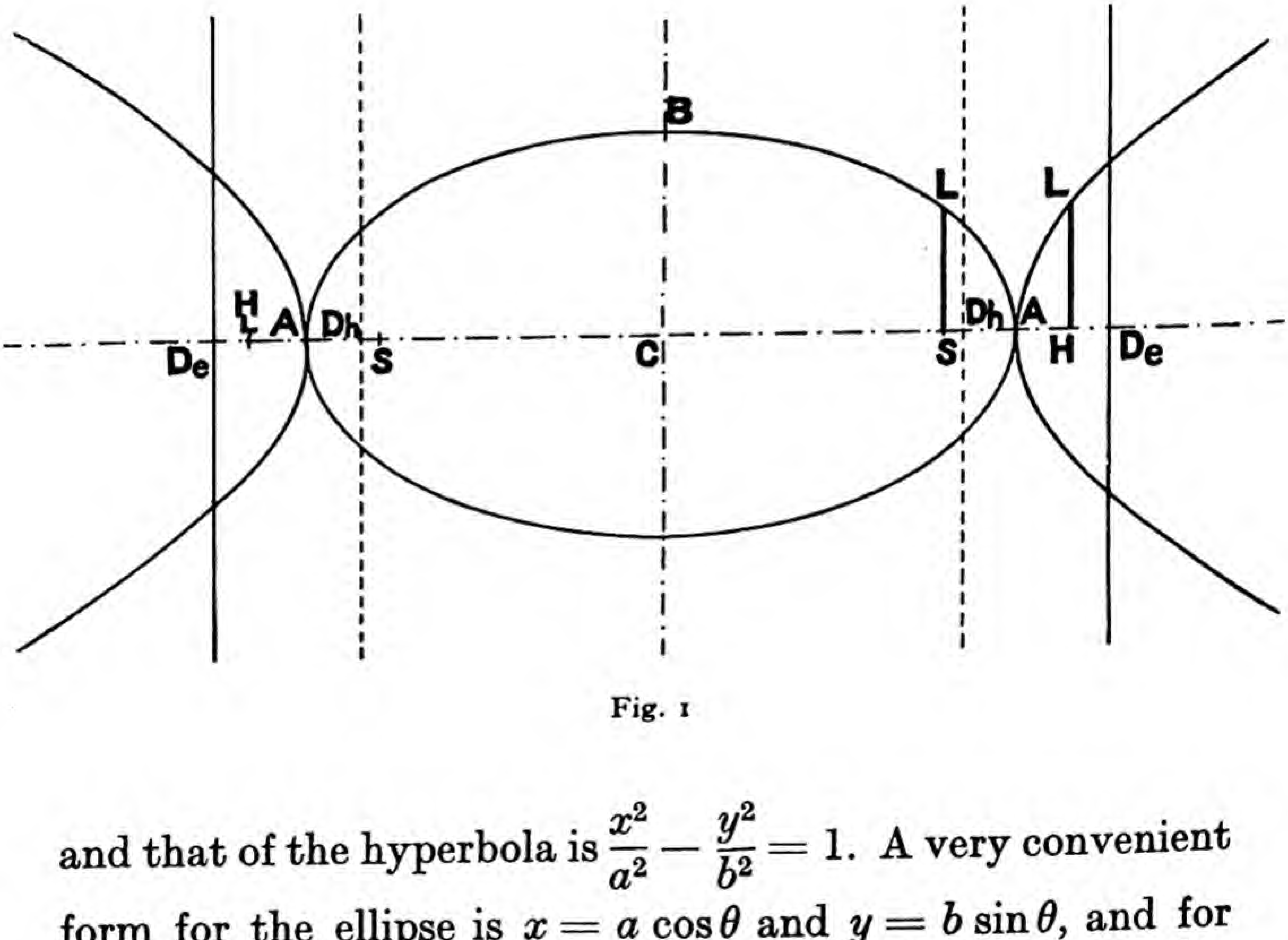

Fig. 1

and that of the hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. A very convenient form for the ellipse is $x = a\cos\theta$ and $y = b\sin\theta$, and for the hyperbola $x = a\sec\theta$ and $y = b\tan\theta$ or $x = a\cosh u$ and $y = b\sinh u$. But other forms often occur in terms of their eccentricity (e), of distances from a focus (S in the diagram for the ellipse, H in that for hyperbola), and of distances from a directrix (D_e and D_h in the diagram). The value of e is given in terms of a and b, and the position of the focus and the directrix in terms of a and e. The numerical values used in the diagram where $a = 5$, $b = 3$, are given in brackets.

Ellipse		Hyperbola	
$e^2 = \dfrac{a^2 - b^2}{a^2}$	$(e = \frac{4}{5})$	$e^2 = \dfrac{a^2 + b^2}{a^2}$	$\left(e = \dfrac{\sqrt{34}}{5} \text{ or } 1{\cdot}166\right)$
$b = CB = a\sqrt{1 - e^2}$	(3)	$b = CB = a\sqrt{e^2 - 1}$	(3)
$AS = a(1 - e)$	(1)	$AH = a(e - 1)$	(·831)
$CS = ae$	(4)	$CH = ae$	(5·831)
$CD = \dfrac{a}{e}$	(6·25)	$CD = \dfrac{a}{e}$	(4·287)
$SL = \dfrac{b^2}{a}$	(1·8)	$HL = \dfrac{b^2}{a}$	(1·8)

B. *When there is no centre*, $b^2 = 4ac$.

Use procedure A (2). As the terms of the second degree must form a perfect square, when xy vanishes either x^2 or y^2 must also vanish, and at the same time possibly x or y.

E.g. (1) $x^2 + 2xy + y^2 - 4\sqrt{2}\,x + 4\sqrt{2}y = 4.$

Here $\tan 2\theta = \infty$, $\theta = 45°$, so operate with $\epsilon^{i\frac{1}{4}\pi}$.
It will be found that the equation reduces to

$$2Y^2 - 8X = 4 \quad \text{or} \quad Y^2 = 4(X + \tfrac{1}{2}),$$

i.e. a parabola with its vertex A at $-\frac{1}{2}$, 0.

(2) Case of parabola in which neither x nor y vanish.

$$x^2 + 2xy + y^2 - 2\sqrt{2}x + 6\sqrt{2}y = 14.$$

On rotating the locus through an angle of 45°, we obtain

$$2Y^2 - 8X + 4Y = 14, \quad \text{or} \quad Y^2 + 2Y = 4X + 7.$$

As the left side of the equation must form a perfect square, the equation is rewritten thus:

$$Y^2 + 2Y + 1 = 4X + 8, \quad \text{or} \quad (Y + 1)^2 = 4(X + 2).$$

The vertex of this parabola is at the point -2, -1.

(3) Case of two parallel lines.

Whenever this procedure results in a quadratic in x or y, two parallel lines are indicated:

$$x^2 + 2xy + y^2 - \sqrt{2}x - \sqrt{2}y = 4.$$

With the same rotation of the locus we obtain

$$(Y - 2)(Y + 1) = 0,$$

i.e. two lines which are parallel to the axis of x in the altered position of the axes.

Polar Equations.

The position of a point can be exactly defined by its distance (r) from an origin or pole at O and the angle (θ) which the line r makes with the initial line passing through O.

Any equation in rectangular co-ordinates can be very easily transformed into a polar equation if the origin or pole remain unchanged, and OX be regarded as the initial line. It is, however, essential to remember that the radius vector (r) may take either a positive or a negative sign.

It is only necessary to write $r\cos\theta$ for x, and $r\sin\theta$ for y; e.g. $b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$ is the polar form for the equation to the ellipse with its origin at the centre, and $b^2r^2\cos^2\theta - a^2r^2\sin^2\theta = a^2b^2$ is the polar equation to the similar hyperbola.

The parabola $y^2 = 4ax$ becomes $r^2\sin^2\theta = 4ar\cos\theta$ or $r\sin^2\theta = 4a\cos\theta$.

CHAPTER IV

Trigonometry and Hyperbolic Functions

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

Let $S = A + B$, and $D = A - B$.

$$\sin S + \sin D = 2 \sin \frac{S+D}{2} \cos \frac{S-D}{2}.$$

$$\sin S - \sin D = 2 \cos \frac{S+D}{2} \sin \frac{S-D}{2}.$$

$$\cos D + \cos S = 2 \cos \frac{S+D}{2} \cos \frac{S-D}{2}.$$

$$\cos D - \cos S = 2 \sin \frac{S+D}{2} \sin \frac{S-D}{2}.$$

$$\sin(x+h) - \sin x = 2 \cos(x + \tfrac{1}{2}h) \sin \tfrac{1}{2}h.$$
$$\cos(x+h) - \cos x = -2 \sin(x + \tfrac{1}{2}h) \sin \tfrac{1}{2}h.$$

Let $m = \frac{1}{2}(S + D)$ and $n = \frac{1}{2}(S - D)$.

$$\sin mx \cos nx = \tfrac{1}{2} \sin(m+n) + \tfrac{1}{2} \sin(m-n)x.$$
$$\sin mx \sin nx = \tfrac{1}{2} \cos(m-n)x - \tfrac{1}{2} \cos(m+n)x.$$
$$\cos mx \cos nx = \tfrac{1}{2} \cos(m-n)x + \tfrac{1}{2} \cos(m+n)x.$$
$$\cos mx \sin nx = \tfrac{1}{2} \sin(m+n)x - \tfrac{1}{2} \sin(m-n)x.$$

$$2 \sin^2 \frac{A}{2} = 1 - \cos A. \qquad 2 \cos^2 \frac{A}{2} = 1 + \cos A.$$

$$\left(\cos \frac{A}{2} - \sin \frac{A}{2}\right)^2 = 1 - \sin A$$

$$\left(\cos\frac{A}{2} + \sin\frac{}{2}\right)^2 = 1 + \sin A.$$

$$\tan\frac{A}{2} = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A}.$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

$$\cos 3A = 4\cos^3 A - 3\cos A.$$
$$\sin 3A = 3\sin A - 4\sin^3 A.$$
$$\cos(n+1)A + \cos(n-1)A = 2\cos nA \cos A.$$
$$\sin(n+1)A + \sin(n-1)A = 2\sin nA \cos A.$$

$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \dots$$

$$\therefore\ \epsilon^{i\theta} = \cos\theta + i\sin\theta.$$
$$\epsilon^{-i\theta} = \cos\theta - i\sin\theta.$$

$$\therefore\ \cos\theta = \tfrac{1}{2}(\epsilon^{i\theta} + \epsilon^{-i\theta}). \qquad \sin\theta = \frac{1}{2i}(\epsilon^{i\theta} - \epsilon^{-i\theta}).$$

Triangles.

1. (*a*) Sine Rule: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}.$

(*b*) $\sin\dfrac{A-B}{2} = \dfrac{a-b}{c}\cos\dfrac{C}{2}.$

2. (*a*) Cosine Rule: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}.$

(*b*) $\cos\dfrac{A-B}{2} = \dfrac{a+b}{c}\sin\dfrac{C}{2}.$

3. Napier's Analogy: $\dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \dfrac{a-b}{a+b}$,

or $$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\tan\frac{A+B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

for $\dfrac{A+B}{2} = 90° - \dfrac{C}{2}$.

4. Semi-sum Rule. Let $s = \frac{1}{2}(a+b+c)$.

$$\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}.$$

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}.$$

$$\tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)}.$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2}{bc}S,$$

where $S = \sqrt{s(s-a)(s-b)(s-c)}$.

Hence the area of a triangle or $\frac{1}{2}bc\sin A = S$.

5. $a = c\cos B + b\cos C$; and from the principle of symmetry which applies to all of the above formulæ it is easily seen that

$$b = a\cos C + c\cos A \quad\text{and}\quad c = b\cos A + a\cos B.$$

Logarithmic Solution of Triangles.

Case (1). Given two angles and any side: say A, B, a. Use Sine Rule.

$$C = 180° - (A+B),\ b = \frac{a}{\sin A}\sin B,\ c = \frac{a}{\sin A}\sin C.$$

Case (2). Given two sides and the included angle: say A, b, c. Use Tangent Rule and 2 (b) Cosine Rule.

(If logarithms are not used, find a by Cosine Rule.)

E.g. $A = 60°$, $b = 4$, $c = 6$.

$$\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2} = \frac{2}{10}\cot 30°.$$

$\cdot 2$	$\bar{1}\cdot 30103$
$\cot 30$	$\cdot 23856$
$\tan\frac{C-B}{2}$	$\bar{1}\cdot 53959$

$$90° - \frac{A}{2} = \frac{C+B}{2} = 60°.$$

$$\frac{C-B}{2} = 19°\ 6'\ 23''.$$

$$C = 79°\ 6'\ 23'',\ B = 40°\ 53'\ 37.''$$

$$\cos\frac{C-B}{2} = \frac{c+b}{a}\sin\frac{A}{2},\ \text{or } a\cos 19°\ 6'\ 23'' = 10\sin 30° = 5.$$

$$a = \frac{5}{\cos 19°\ 6'\ 23''}.$$

5	$\cdot 69897$
$\cos 19°\ 6'\ 23''$	$\bar{1}\cdot 97539$
$a = 5\cdot 2915 = AL$	$\cdot 72358$

AL denotes antilogarithm.

Time will be saved by noting the value of $\sin\frac{1}{2}A$ when determining $\cot\frac{1}{2}A$, and similarly with $\cos\frac{1}{2}(C - B)$ and $\tan\frac{1}{2}(C - B)$.

Case (3). Given two sides and the angle opposite one of them: say A, a, b. Use Sine Rule.

$$\sin B = \frac{b}{a}\sin A. \qquad C = 180° - (A + B).$$

$$c = a\frac{\sin C}{\sin A} \quad \text{or} \quad b\frac{\sin C}{\sin B}.$$

Now since $\sin B$ has the same value as $\sin(180 - B)$, clearly in some cases B may be either acute or obtuse; the solution is then said to be *ambiguous*. The ambiguous case can only arise if A be acute, and if a lie between b and $b\sin A$ in magnitude.

Case (4). Given three sides: a, b, c.

If the lengths (a, b, and c) are denoted by numbers of only one or two figures, it is usually speedier not to use logarithms, but to use the Cosine Rule, 2 (a).

If the numbers consist of many figures, the tangent

form in (4) is the best, as only four logarithms are required for any angle.

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}.$$

$$\tan\frac{B}{2} = \sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}}.$$

$C = 180° - (A + B)$.

Two Trigonometrical Forms useful in Algebra.

The arithmetical root of $\sqrt{4}$ is 2, of $\sqrt[3]{8}$ is 2; but if a sense of direction (i.e. a vectorial meaning) is required, $\sqrt{4}$ has two roots ± 2, and $\sqrt[3]{8}$ has three roots. Similarly there are n values of $\sqrt[n]{a^n}$, though a is the only arithmetical root.

Let a be the length of a radius vector OA rotating counter-clockwise round a centre O; when it has been rotated half a revolution (π) it will occupy a reversed position OA' and will be denoted by $-a$, and on again repeating this rotation of π it will have regained its primary position of OA or a. This is the meaning of $\sqrt{a^2}$ having two roots $\pm a$.

Similarly if a be the arithmetic nth root of a^n, there will be n vectorial roots—one, a, when OA is in the primary position, another when it has rotated $\frac{2\pi}{n}$, and indeed others when it has rotated $\frac{m2\pi}{n}$, where m takes every integral value from 0 to n, when it regains its original position as $+a$. We have then merely to operate on the vector a with the versor $\epsilon^{\frac{im2\pi}{n}}$ or $\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}$, giving m every integral value from 0 to $n-1$.

Thus to find the three cube roots of unity, or $\sqrt[3]{1}$, we have

$$1,\quad \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right),\quad \left(\cos\frac{2(2\pi)}{3} + i\sin\frac{2(2\pi)}{3}\right),$$

or $$1,\quad \left(-\tfrac{1}{2} + \frac{\sqrt{-3}}{2}\right),\quad \left(-\tfrac{1}{2} - \frac{\sqrt{-3}}{2}\right).$$ (See Tables, p. 115.)

Factors of $x^n \pm a^n$ where n is integral.

$x^n + a^n$. The expression $x^2 + a^2$ has no real factors. If n be odd, one factor is $(x + a)$; the other factors are of the form $\left(x^2 - 2ax\cos\frac{r\pi}{n} + a^2\right)$, where r takes all *odd* integral values $< n$. If n be even, all the factors are quadratics of this form, r taking all *odd* integral values $< n$.

$$x^3 + a^3 = (x + a)\left(x^2 - 2ax\cos\frac{\pi}{3} + a\right)$$
$$= (x + a)(x^2 - ax + a^2).$$

$$x^4 + a^4 = \left(x^2 - 2ax\cos\frac{\pi}{4} + a^2\right)\left(x - 2ax\cos\frac{3\pi}{4} + a^2\right)$$
$$= (x^2 - ax\sqrt{2} + a^2)(x + ax\sqrt{2} + a^2).$$

$$x^5 + a^5 = (x + a)\left(x^2 - ax\frac{\sqrt{5} + 1}{2} + a^2\right)$$
$$\left(x^2 + ax\frac{\sqrt{5} - 1}{2} + a^2\right).$$

$$x^6 + a^6 = (x^2 - ax\sqrt{3} + a^2)(x^2 + a^2)(x^2 + ax\sqrt{3} + a^2),$$

and so on. (See Tables, p. 115.)

$x^n - a^n$. The form $x^2 - a^2 = (x - a)(x + a)$. In other cases if n be even the general form is

$$(x^2 - a^2)\left(x^2 - 2ax\cos\frac{r\pi}{n} + a^2\right),$$

where r takes all the *even* integral values $< n$. If n be odd the first factor is $(x - a)$, and the other quadratic factors are as before, r taking all the *even* integral values $< n$.

$$x^3 - a^3 = (x - a)\left(x^2 - 2ax\cos\frac{2\pi}{3} + a^2\right)$$
$$= (x - a)(x^2 + ax + a^2).$$

$$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2).$$

$$x^5 - a^5 = (x-a)\left(x^2 - 2ax\cos\frac{2\pi}{5} + a^2\right)\left(x^2 - 2ax\cos\frac{4\pi}{5} + a^2\right)$$

$$= (x-a)\left(x^2 - ax\frac{\sqrt{5}-1}{2} + a^2\right)\left(x + ax\frac{\sqrt{5}+1}{2} + a^2\right).$$

$$x^6 - a^6 = (x^2 - a^2)(x^2 - ax + a^2)(x^2 + ax + a^2).$$

$$x^8 - a^8 = (x^2 - a^2)(x^2 - ax\sqrt{2} + a^2)(x^2 + a^2)(x^2 + ax\sqrt{2} + a^2).$$

(See Tables, p. 115.)

Hyperbolic Functions.

These are analogous to trigonometrical functions.

E.g. $$\cosh^2 u - \sinh^2 u = 1.$$

$$\sinh(u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v.$$
$$\cosh(u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v.$$

$$2\cosh^2\frac{u}{2} = \cosh u + 1. \qquad 2\sinh^2\frac{u}{2} = \cosh u - 1.$$

$$\tanh\frac{u}{2} = \frac{\cosh u - 1}{\sinh u} = \frac{\sinh u}{1 + \cosh u}, \text{ \&c.}$$

$$\cosh u = 1 + \frac{u^2}{\underline{|2}} + \frac{u^4}{\underline{|4}} + \ldots = \tfrac{1}{2}(\epsilon^u + \epsilon^{-u}).$$

$$\sinh u = u + \frac{u^3}{\underline{|3}} + \frac{u^5}{\underline{|5}} + \ldots = \tfrac{1}{2}(\epsilon^u - \epsilon^{-u}).$$

$$\therefore\ e^u = \cosh u + \sinh u, \quad e^{-u} = \cosh u - \sinh u.$$

If θ be the gudermannian of u, all the hyperbolic functions can be given in terms of $\mathrm{gd}\,u$ or θ.

$$\sinh u = \tan\theta. \quad \mathrm{cosech}\, u = \cot\theta.$$
$$\cosh u = \sec\theta. \quad \mathrm{sech}\, u = \cos\theta.$$
$$\tanh u = \sin\theta. \quad \coth u = \mathrm{cosec}\,\theta.$$

Also $$\tanh\frac{u}{2} = \tan\frac{\theta}{2},$$

for $$\frac{\cosh u - 1}{\sinh u} = \frac{\sec\theta - 1}{\tan\theta} = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}.$$

The Gudermannian.—If a hyperbolic function be given, the corresponding circular function can be written down at once from the tables.

If u be given, this is the procedure.

$$\text{As } e^{-u} = \cosh u - \sinh u = \sec\theta - \tan\theta$$

$$= \frac{1-\sin\theta}{\cos\theta} \text{ or } \frac{1-\cos A}{\sin A} \text{ when } A = \frac{\pi}{2} - \theta,$$

$$e^{-u} = \frac{2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A} = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right).$$

$$\therefore\ e^{u} = \cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

As an example let $u = \cdot 516$, or $\lg e^{\cdot 516} = \cdot 516$.

Use the Conversion Tables (p. 112) to find $\log e^{\cdot 516}$ to 8 decimal places at least to obtain θ to the nearest second.

$$\begin{array}{r} \cdot 217147241 \\ \cdot 004342945 \\ \cdot 002605767 \\ \hline \end{array}$$

$$\log e^{\cdot 516} = \cdot 224095953 = \log\tan 59^\circ\ 10'\ 0{\cdot}76''.$$

$$e^{u} \text{ or } \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan(45^\circ + 14^\circ\ 10'\ 0{\cdot}76'').$$

$$\therefore\ \mathrm{gd}\,u \text{ or } \theta = 28^\circ\ 20'\ 1{\cdot}5''.$$

Inverse Hyperbolic Functions.

$$\sinh^{-1}\frac{mx}{a} = \lg\frac{mx + \sqrt{m^2x^2 + a^2}}{a}. \qquad \mathrm{cosech}^{-1}\frac{mx}{a} = \lg\frac{a + \sqrt{a^2 + m^2x^2}}{mx}.$$

$$\cosh^{-1}\frac{mx}{a} = \lg\frac{mx + \sqrt{m^2x^2 - a^2}}{a}. \qquad \mathrm{sech}^{-1}\frac{mx}{a} = \lg\frac{a + \sqrt{a^2 - m^2x^2}}{mx}.$$

$$\tanh^{-1}\frac{mx}{a} = \tfrac{1}{2}\lg\frac{a + mx}{a - mx}. \qquad \coth^{-1}\frac{mx}{a} = \tfrac{1}{2}\lg\frac{mx + a}{mx - a}.$$

CHAPTER V

Finite Differences, Graphs, Observational Equations, General Equations

Finite Differences.

Let there be a series of numbers, say u_0, u_1, u_2, u_3, u_4, &c., where $u_0 = F(x)$, $u_1 = F(x+h)$, $u_2 = F(x+2h)$, &c.; let $\Delta_1 u_0 = u_1 - u_0$, $\Delta_1 u_1 = u_2 - u_1$, &c., and in a similar way take a second, third, or fourth order of differences until their value becomes 0 or negligible. Expressions will be given for the nth order of differences in terms of the values of u, for the general term of the series, and for the sum of the series to n terms when that is possible. An example will make the procedure plain.

$$\begin{array}{llccc}
 & & \Delta_1 & \Delta_2 & \Delta_3 \\
u_0 & 2 & & & \\
 & & 1 & & \\
u_1 & 3 & & 8 & \\
 & & 9 & & 0 \\
u_2 & 12 & & 8 & \\
 & & 17 & & 0 \\
u_3 & 29 & & 8 & \\
 & & 25 & & \\
u_4 & 54 & & &
\end{array}$$

Here $\Delta_1 u_0 = 1$, $\Delta_2 u_0 = 8$, $\Delta_3 u = 0$, and all subsequent orders of differences vanish also. It must be noticed that $u\Delta_1$ has no precise meaning; Δ is an operator, and the operand u on which it is to operate must be placed after the symbol Δ to indicate this.

It will be found that the *symbolical* expression

$$u_n = (1 + \Delta)^n u_0 \quad \text{or} \quad u_0 + n\Delta_1 u_0 + \frac{n.\overline{n-1}}{\underline{|2}} \Delta_2 u_0 + \ldots$$

will give the numerical value of u_n, if we agree to change the exponents of the powers of Δ into suffixes to indicate the corresponding orders of differences.

E.g. $\quad u_4 = u_0 + 4\Delta_1 u_0 + 6\Delta_2 u_0 = 2 + 4 + 48 = 54.$

$$u_n = 2 + n + \tfrac{1}{2}n(n-1)8, \quad \text{or} \quad 4n^2 - 3n + 2$$

is the *general* term of the above sequence.

The nth order of differences is $\Delta_n u_0 = (u - 1)^n$ if we change the exponents of the powers of u into suffixes to indicate the corresponding orders of the terms. Note that instead of 1 for the last term one must suppose that its equivalent u_0 is substituted.

E.g. $\Delta_2 u_0 = (u_0 - 1)^2 = u_2 - 2u_1 + u_0 = 12 - 6 + 2 = 8.$

Now for the sum of n terms of the series the summation symbol Σ or Δ^{-1} is used which neutralizes Δ_1 in the expressions that follow it. The sum of n terms of the series is

$$\Delta^{-1}\{(1 + \Delta)^n - 1\}u_0$$

$$= \Delta^{-1}\left\{n\Delta_1 u_0 + \frac{n.\overline{n-1}}{\underline{|2}}\Delta_2 u_0 + \frac{n.\overline{n-1}.\overline{n-2}}{\underline{|3}}\Delta_3 u_0 + \ldots\right\}$$

$$= nu_0 + \frac{n.\overline{n-1}}{\underline{|2}}\Delta_1 u_0 + \frac{n.\overline{n-1}.\overline{n-2}}{\underline{|3}}\Delta_2 u_0 + \ldots$$

E.g. the sum of 5 terms of the previously given series is

$$\Delta^{-1}\{(1 + \Delta)^5 - 1\}u_0 = 5(2) + 10(1) + 10(8) = 100,$$

as all the differences beyond the second order vanish. It will be noted that if any constant term, say 6, be added to all the terms of the series, e.g. 8, 9, 18, 35, 60, &c., the values of the three orders of differences will remain un-

changed. Hence when the orders of differences are given alone, the sum of the series is expressed by

$$\Sigma\{(1+\Delta)^n - 1\}u_0 = nu_0 + \frac{n\,.\,\overline{n-1}}{\underline{|2}}\,\Delta_1 u_0 + \&c. + C,$$

where C is any constant positive or negative. If the arbitrary constant is not required, it is well to use the symbol Δ^{-1}.

Interpolation.—This is effected by using the formula for u_n, or by using the general term of the sequence. In either case $u_{3\frac{1}{3}}$, for instance, is found to be $36\frac{4}{9}$.

Graphs.

Nearly all experiments in a laboratory are made to determine how one result y depends upon a condition x. It is well in every case to plot the results on squared paper, and note what the curve suggests.

(1) If the curve be nearly a straight line, $y = mx + c$ is suggested.

(2) If the rate of increase or decrease $\left(\frac{dy}{dx}\right)$ is proportional to the function itself (y), $y = ae^{bx}$ is suggested.

(3) If $\frac{dy}{dx}$ be proportional to x, try $y = a + bx^2$.

(4) If the curve alternately rises and falls, some trigonometrical function is suspected, perhaps $y = a\sin(bx + c)$ or $a\sin(b\theta + \phi)$.

If the oscillations gradually diminish in amplitude, some resistance is being encountered; try $y = ae^{-kx}\sin(bx + c)$.

The angles θ and ϕ (or x and c) are usually measured in radians, so for practice it is well to take c or ϕ as $\frac{\pi}{6} = \cdot 5236$ or any multiple of this, and b to be $\frac{1}{114\cdot 6}$, which is practically $\frac{1}{2}°$, so if $a = 5$, $x = 6$, and $y = a\sin(bx + c)$,

$$y = 5\sin\left(\frac{6}{114\cdot 6} + \cdot 5236\right) \text{ or } 5\sin(\tfrac{1}{2}6° + 30°) = 5\sin 33°.$$

This is what Perry recommends, and he says that in engineering problems the three most frequently recurring formulæ are

$$y = ax^n, \quad y = a\sin(bx + c), \quad y = ae^{bx}.$$

Another formula he gives, $y = \dfrac{ax}{1 + bx}$, which is $\dfrac{y}{x} + by = a$; find $\dfrac{y}{x}$ and call it X, plot $X + by = a$; if the points lie on a straight line some such law as $y = \dfrac{ax}{1 + bx}$ is suggested; b may be either positive or negative.

Of course we are only dealing with empirical formulæ formed entirely from the observations made, and these must never be regarded as theoretical formulæ which are mathematically deduced from known principles and known physical laws.

We may also try the expression $y = a + bx + cx^2 + \ldots$.

Observational Equations.

We often deal with cases in which we are led to try whether an equation such as $y = A + Bx + Cx^2 + \ldots$ or $y = A + Bu + Cv + \ldots$ may be used to express our instrumental observations. Here u, v, &c., are expressed in numerical units of time, volume of gas evolved, voltage, or any condition that may exist during the observation. Similarly x is known; it may represent a distance, a velocity, or anything else that can be measured.

We may make twenty experiments and yet no one of them will be measured really accurately, nor will the equations expressing their results be consistent with each other. The results obtained will be denoted by r, and from our n observations we must find three Normal equations which are consistent with each other to determine the unknown constants A, B, and C. Of course as many Normal equations will be required as there are unknown constants to determine.

Least squares. Let v denote the error $(y - r)$ which gives the difference between the result obtained and the true result if our

supposed equation were the correct one. The object is to make the sum of the squares of the errors a minimum, or $\Sigma(v^2)$ a minimum.

Take $y = A + Bx + Cx^2$ as an example, then $\Sigma(A + Bx + Cx^2 - r)^2$ must be a minimum.

Partially differentiate the expression

$$\Sigma(A^2 + 2ABx + 2ACx^2 - 2Ar + B^2x^2 + 2BCx^3 - 2Bxr + C^2x^4 - 2Cx^2r + r^2)$$

with regard to A, B and C, and equate the result to zero, in order to find the best values for these unknown constants that will make $\Sigma(v^2)$ a minimum.

$$\frac{\partial}{\partial A}\Sigma(A + Bx + Cx^2 - r)^2 = 0. \quad \therefore\ 2\Sigma(A + Bx + Cx^2 - r) = 0.$$

$$\frac{\partial}{\partial B}\Sigma(A + Bx + Cx^2 - r)^2 = 0. \quad \therefore\ 2\Sigma x(A + Bx + Cx^2 - r) = 0.$$

$$\frac{\partial}{\partial C}\Sigma(A + Bx + Cx^2 - r)^2 = 0. \quad \therefore\ 2\Sigma x^2(A + Bx + Cx^2 - r) = 0.$$

The constants A, B, and C that we have to determine are here regarded as variables, and for convenience are distinguished by capital letters. From the equations just given it will be seen that the following rules, called Gauss's rules, arise, but when giving them it will be well to supply an ample.

The figures obtained by experiment are given in the adjoining table, and it is suspected that they might be due to the law $y = A + Bx + Cx^2$.

x	0	1	2	3
r	4·9	10	20·9	39

(1) Write out all the results obtained in accordance with this law, and add them together. This will give the first Normal equation, i.e. the equation obtained above by differentiating with respect to A.

$$\begin{aligned} A \qquad\qquad\quad &= 4.9 \\ A + B + C &= 10 \\ A + 2B + 4C &= 20{\cdot}9 \\ A + 3B + 9C &= 39 \\ \hline (1) \quad 4A + 6B + 14C &= 74{\cdot}8 \end{aligned}$$

(2) Multiply each equation by the coefficient of B in it, i.e. by the value of x in it, and then add the equations together. This will give the second Normal equation.

$$\begin{array}{rl} & A + B + C = 10 \\ & 2A + 4B + 8C = 41{\cdot}8 \\ & 3A + 9B + 27C = 117 \\ \hline (2) & 6A + 14B + 36C = 168{\cdot}8 \end{array}$$

(3) Multiply each equation by the coefficient of C in it, i.e. by x^2, and then add the equations together, and the third Normal equation will be obtained.

$$\begin{array}{rl} & A + B + C = 10 \\ & 4A + 8B + 16C = 83{\cdot}6 \\ & 9A + 27B + 81C = 351 \\ \hline (3) & 14A + 36B + 98C = 444{\cdot}6 \end{array}$$

On using the determinant method described on p. 15, it is found that to eliminate B and C the first equation must be multiplied by 76, the second by -84, and the third by 20. Finally, A is found to be 4·97, $B = 1{\cdot}59(3)$, $C = 3{\cdot}24$, and the sum of the squares of the errors is 0·0971.

If the equation suggested were $y = A + Bu + Cv$, &c., the procedure would be the same. The symbols u and v are represented by definite numbers for each experiment, which are used as coefficients of the unknown constants B and C. The first Normal equation is formed by adding all the equations together, the second by multiplying each equation by the coefficient of B in it and then adding together these equations so altered, the third similarly by using the coefficient of C.

General Equations.

In practical work observations are made under certain observed conditions; we wish to formulate the *general law* governing the relation of the conditions to the result obtained. A simple example will illustrate the principle of

the method. Suppose that two motor cars A and B are travelling in the same direction, each with a constant velocity, A with a speed of 30 m.p.h., B with a speed of 20 m.p.h. At noon A is 10 miles behind B; when will A be 15 miles in front of B?

Fig. 2

Draw a rough diagram of the conditions obtaining. Write down what you infer from the diagram, using *geometrical* reasoning only, i.e. let AB or BA represent a length only without any sense of direction.

In time T, A will have travelled AA' or T 30 miles, and B will have travelled BB' or T 20 miles; $AB = 10$ miles and $B'A' = 15$ miles.

From the diagram it is seen that

$$AA' = AB + BB' + B'A' \quad \text{or} \quad T\,30 = 10 + T\,20 + 15,$$
$$\text{or} \quad T(30 - 20) = 25. \quad T = 2{\cdot}5 \text{ hr. or } 2 \text{ hr. } 30 \text{ min.}$$

This gives only the special case considered, but we wish to find the general equation that will give us the time when B and A are separated by a given distance with either B or A in advance, and when the cars are travelling in opposite directions.

To obtain the *general* equation we must introduce the sense of direction into our symbols, i.e. they must be regarded as vectors. Both velocities and the final separation $B'A'$ in the diagram happen to be in the positive direction, i.e. from left to right. As it is natural to regard the faster car A in front of the slower car, we will denote $B'A'$ by D and BA by d_0, so AB will be $-d_0$; V denotes the faster and v the slower velocity.

We have then from the diagram

$$AA' = AB + BB' + B'A'$$

or
$$TV = -d_0 + Tv + D,$$

i.e.
$$D = d_0 + T(V - v) \text{ for the general equation.}$$

By altering the sign of any one of these symbols the direction will be reversed. If D be negative $(A'B')$, and d_0 be also negative as before, we have $-15 + 10 = T(30 - 20)$:

$$T = -\tfrac{5}{10} = -\tfrac{1}{2} \text{ hr., or at 11 hr. 30 min.}$$

If the cars are travelling in opposite directions, we have merely to alter the sign of the velocity which is in the negative direction. This *general* equation will deal with any of the sixteen conditions that may arise, when one or more of the terms (d, D, V, v) has its sign changed.

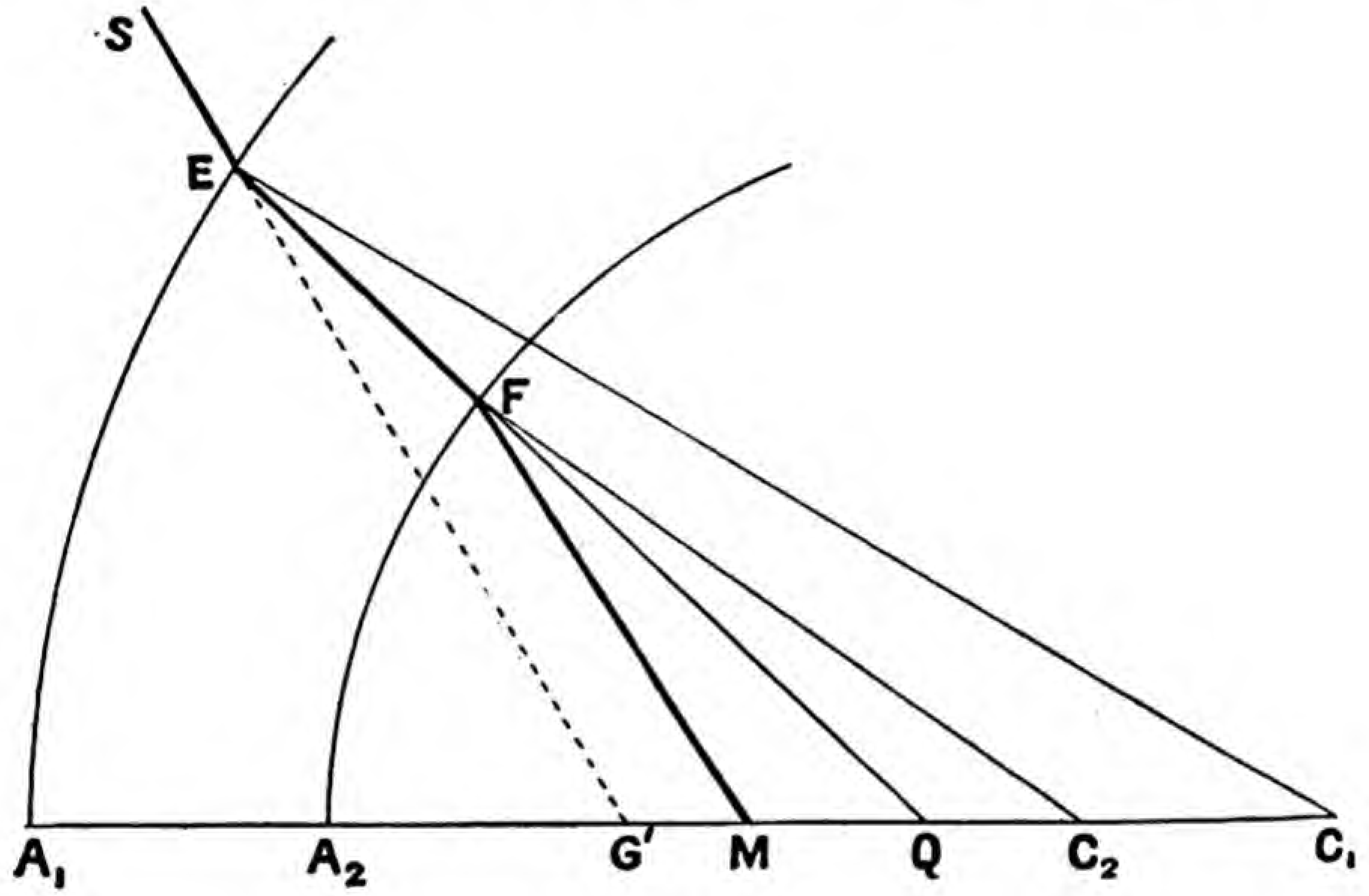

Fig. 3

The complete rules will now be given which will deal with more complicated cases; they must be taken in order.

Take, as an example, a problem that arises in the design of periscopic spectacles. The eye, which admits only a thin incident pencil through its pupil, is allowed to rotate an angle θ (say 30°) about its centre of rotation (M) behind the spectacle-lens; the refracted image of a point at the macula should not have a circle of confusion larger than the retinal surface of a macular cone.

It is found that the ocular surface (A_2F) of the lens must be concave, the power of the lens required is known, and

we will suppose that r_1 and r_2, the radii of curvature of the first and second surfaces of the lens, are known and also its axial thickness t (i.e. A_1A_2). It is necessary that the axis of the refracted pencil should pass through M, and A_2M or k is also known. We wish to find the angles of incidence and refraction ϕ, ϕ' at the first surface of the lens, and also ψ', ψ, those at the second surface, in terms of the known values.

I. *Make a rough diagram of the simplest case* (fig. 3). Indicate the symbols used by the letters in the diagram, separating the known from the unknown. For instance in this diagram:

KNOWN	UNKNOWN
$t = A_1A_2.$	$\phi = C_1EG'.$
$k = A_2M.$	$\phi' = C_1EQ.$
$r_1 = C_1A_1$ or $C_1E.$	$\psi = C_2FM.$
$r_2 = C_2A_2$ or $C_2F.$	$\psi' = C_2FQ.$
$\theta = A_2MF.$	

II. *Write down what you infer from the diagram*, paying no attention to the direction of the lengths or of the angles, i.e. regard $C_2M = MC_2$ as simply a length and $FMC_2 = C_2MF$ an angle without any indication of its direction as in Euclid. Use any trigonometrical expression that may be needed, differentiate it if necessary, and so transform the expression that it will give the result you require. The formula obtained in this way will only apply to the special case instanced in the diagram.

(1) To find C_2FM:

$$\frac{\sin C_2FM}{\sin FMC_2} = \frac{C_2M}{FC_2} = \frac{C_2A_2 - A_2M}{FC_2};$$

and as $\sin FMC_2 = \sin A_2MF$,

$$FC_2 \sin C_2FM = (C_2A_2 - A_2M) \sin A_2MF.$$

(2) To find C_1EQ:

$$(a)\ \frac{\sin C_1EQ}{\sin EQC_1} = \frac{C_1Q}{EC_1}, \quad (b)\ \frac{\sin C_2FQ}{\sin FQC_2} = \frac{C_2Q}{FC_2}.$$

As $\sin EQC_1$ or $\sin FQC_2 = \sin A_2QF$,

$$EC_1 \sin C_1EQ = C_1Q \sin A_2QF.$$
$$FC_2 \sin C_2FQ = C_2Q \sin A_2QF.$$

On subtracting (*b*) from (*a*) we obtain

$$EC_1 \sin C_1EQ - FC_2 \sin C_2FQ = (C_1Q - C_2Q) \sin A_2QF$$
$$= (C_1A_1 - A_1A_2 - A_2C_2) \sin (A_2MF - C_2FM + C_2FQ).$$

We have now to find the *general* formula in terms of the symbols which all have an *assigned direction*. On looking at the angles θ, ϕ, &c., we see that in the diagram they will all be measured in the clockwise direction, i.e. in the negative direction, and the radii also.

III. *All the lengths and all the angles must be measured in one direction, either positive or negative.*

As most of the symbols are measured in the negative direction, we shall choose the negative direction, and formulæ (1) and (2) must be rewritten.

(1) becomes $C_2F \sin C_2FM = (C_2A_2 - MA_2) \sin A_2MF$;

(2) becomes $C_1E \sin C_1EQ - C_2F \sin C_2FQ$
$$=(C_1A_1 - A_2A_1 - C_2A_2) \sin(A_2MF - C_2FM + C_2FQ).$$

IV. *Now introduce the symbols which have a fixed direction so as to represent the lettered symbols thus altered.*

(1) is $\quad r_2 \sin\psi = (r_2 + k) \sin\theta$;

(2) is $\quad r_1 \sin\phi' - r_2 \sin\psi' = (r_1 + t - r_2) \sin(\theta - \psi + \psi')$.

As $\sin\phi = \mu \sin\phi'$ and $\sin\psi = \mu \sin\psi'$, all the required angles can now be found from these formulæ, which will give correct solutions if all the known symbols are either positive or negative; if some of them are of contrary sign, these must be altered accordingly. In fact, they both are *general* formulæ which can be applied to any similar case whether the power of the lens be positive or negative.

CHAPTER VI

Differentiation

When one quantity (y) depends upon another or upon a system of others for its value, y is said to be a function of these others. Thus $y = f(x)$ or $F(u, v)$ means that as different values are assigned to the independent variables (x, u, v, or any other "arguments" of the function) the dependent variable y assumes a definite value.

The differential calculus deals primarily with the measurements of the rates of change of y as compared with the changes in the independent variable or argument.

If $y = f(x)$ represents a curve, the tangent at a given point (x, y) on the curve will represent the rate of growth of y as compared with the growth of x. The tangent of the angle which this tangent line makes with the axis of x will denote this ratio, which is expressed by $\frac{dy}{dx}$ or $f'(x)$.

The sign of the derivative tells whether $f(x)$ is increasing or diminishing. If x is increasing through a, $f(x)$ or y increases if $f'(a)$ is positive, but decreases if $f'(a)$ is negative; when $f'(a) = 0$, the tangent at that point is horizontal, and for the moment the value of $f(a)$ is stationary, and it usually has a turning value.

Tangents and Normals of the Curve $y = f(x)$.

Let X and Y be the current co-ordinates of any point on the tangent line, then $Y - y = m(X - x)$ is a line that passes through the point x, y on the curve, and if $m = \frac{dy}{dx}$ or $f'(x)$ at the required point x, y, the equation to the tangent

to the curve at that point is given; e.g. $y^2 = 25 - x^2$ denotes a circle on which the tangent at the point $x = 3$, $y = 4$ must be expressed by an equation. Now

$$2y\frac{dy}{dx} = -2x, \text{ or } \frac{dy}{dx} = -\frac{x}{y}, \text{ or } -\tfrac{3}{4}$$

at the required point.

Then $Y - 4 = -\frac{3}{4}(X - 3)$ or $Y = -{\cdot}75X + 6{\cdot}25$ is the tangent to the circle at this point.

The normal at this point must be at right angles to the tangent, so if $\tan\theta = -{\cdot}75$, $\tan\left(\frac{\pi}{2} + \theta\right)$ or $-\cot\theta$ must be the value of m' for the normal. But $-\cot\theta = -\frac{1}{\tan\theta} = \frac{4}{3}$, or $m' = -\frac{1}{m}$.

The equation to the normal at a point x, y is then

$$Y - y = -\frac{1}{m}(X - x) \text{ or } \frac{dy}{dx}(Y - y) + X - x = 0.$$

In the above case $Y - 4 = \frac{4}{3}(X - 3)$ or $3Y = 4X$. The normal must therefore pass through the origin; in this case it is the radius of the circle to the point.

Curvature.

Let normals be drawn from two adjacent points P and Q on a plane curve; these normals will intersect at a point C. The angle between these normals subtended by the arc PQ (s) at C is the angle (ψ) of deflection of the tangent at P to the tangent at Q. When Q approaches P as its limiting position, $\frac{d\psi}{ds}$ is called the *curvature* at the point P, and is the rate of deflection of the tangent at that point. The angle ψ is always measured in radians. The curvature of a circle is the same at every point, and is measured by the reciprocal of the radius, $\frac{d\psi}{ds} = \frac{1}{r}$.

In any curve it is usual to denote the radius of curva-

ture at the point P by ρ or $\frac{ds}{d\psi}$, and the curvature by $\frac{1}{\rho}$, C being called the centre of curvature at that point. The form $\rho = \frac{ds}{d\psi}$ can rarely be used except in intrinsic equations, so two other forms are given.

For Cartesian equations $\rho = \dfrac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$.

The curvature is $\frac{1}{\rho}$ or $\frac{d^2y}{dx^2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-\frac{3}{2}}$; or, writing Dy for $\frac{dy}{dx}$, and D^2y for $\frac{d^2y}{dx^2}$,

$$\frac{1}{\rho} = D^2y\left[1 - \tfrac{3}{2}(Dy)^2 + \frac{\frac{3}{2}\,\frac{5}{2}}{\underline{|2}}(Dy)^4 - \ldots\right],$$

so that when Dy is very small $\frac{1}{\rho} \approx D^2y$ or $\frac{d^2y}{dx^2}$. This is a useful form when bending of beams is considered.

For p, ψ equations, $\rho = p + \frac{d^2p}{d\psi^2}$.

Maxima and Minima.

A " maximum " value of a continuous function is one that is greater, and a " minimum " one that is less, than the values in the immediate neighbourhood of the point considered. The necessary and sufficient condition that $f(a)$ may be a maximum value of $f(x)$ is that $f'(x)$ should change sign from $+$ to $-$ as x increases through the value a, usually through 0. It is possible that the tangent at the point $x = a$ at which a maximum or a minimum occurs may be either horizontal or vertical. If the tangent be vertical, or $f'(a) = \infty$, the point a will probably be a cusp; but cusps, nodes, and "conjugate points" will rarely if ever

occur in practical work. We here consider only cases in which $f'(a) = 0$, i.e. when the tangent is horizontal.

The second requisite is satisfied by the condition that the second derivative does not vanish, or if it does, that the *first non-vanishing* derivative is of an even order. If such a derivative be *negative*, $f(a)$ is a *maximum*; if it be *positive*, $f(a)$ is a *minimum* when $x = a$. To show the procedure this example is given.

$$f(x) = 3x^4 - 8x^3 + 6x^2 + 4.$$
$$f'(x) = 12x^3 - 24x^2 + 12x = (12)x(x-1)^2.$$

The factor 12 may be omitted; clearly $f'(x) = 0$ when $x = 0$ and when $x = 1$.

$$f''(x) = 3x^2 - 4x + 1.$$

Now give the above values to x in this equation; if x be replaced by 0, the value of $f''(0)$ is $+1$, therefore $f(x)$ has a *minimum* value when $x = 0$. $f(0)$ is then $+4$.

If $f''(x)$ be given the value $+1$, $f''(1) = 0$, so we must try the higher derivatives; $f'''(x) = 6x - 4$, which when x is given the value 1 is equal to $+2$. The value $+1$ for x is neither a maximum nor a minimum, for $f'''(x)$ is an odd derivative. In all cases in which the first two derivatives vanish and the third has a significant value the point is called a point of inflection. The tangent at this point is horizontal; if the sign of the third derivative is positive, the tangent is above the curve immediately to the left of the point of inflection, and below the curve immediately to the right, and vice versa if the sign is negative.

Again, if the third derivative vanish, and if the fourth derivative do not vanish, there is a point of " undulation " where the curve does not cross the tangent; in this case a maximum or a minimum is distinguished by its sign.

On p. 56 will be found a list of some cases in which the nth derivative can be at once written down.

The following three theorems are of outstanding importance.

Leibnitz's Theorem.

Leibnitz's theorem gives the nth derivative of a product of two functions of x, (u and v), in terms of the derivatives of the separate functions.

Let D_0 operate only on u, and D_1 only on v, then

$$D(uv) = vD_0u + uD_1v.$$

Then $D^n(uv) = (D_0 + D_1)^n\,(uv)$

$$= \left(D_0{}^n + nD_0{}^{n-1}D_1 + \frac{n(n-1)}{\underline{|2}}D_0{}^{n-2}D_1{}^2 + \ldots + D_1{}^n\right)uv$$

$$= (D_0{}^n u)v + nD_0{}^{n-1}uD_1v + {}_nC_2 D_0{}^{n-2}uD_1{}^2v + \ldots + uD_1{}^n v.$$

This form is often useful in finding the successive derivatives of a product; e.g.

$D^2(x^2 \sin ax)$

$$= 2\sin ax + 4xa\sin\left(ax + \frac{\pi}{2}\right) + x^2a^2\sin\left(ax + \frac{2\pi}{2}\right).$$

Taylor's Theorem.

If a function $f(x+h)$ can be expanded in ascending integral positive powers of h, it will be found to take this form:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}}f''(x) + \frac{h^3}{\underline{|3}}f'''(x) + \ldots$$

Taylor's theorem may be symbolically written

$$f(x+h) = \left(1 + hD + \frac{h^2}{\underline{|2}}D^2 + \frac{h^3}{\underline{|3}}D^3 + \ \ldots\ \right)f(x)$$

or
$$f(x+h) = e^{hD}f(x).$$

This will be found a very suggestive form when dealing with partial differential equations with constant coefficients.

The expansion fails or becomes unintelligible if:

(1) $f(x)$ or one of its derivatives becomes infinite between the values of the variable considered, i.e. x and $x+h$.

(2) $f(x)$ or one of its derivatives becomes discontinuous between these values.

There is another form of this theorem which was first given by Stirling, but it is almost always called Maclaurin's theorem.

Maclaurin's Theorem.

If $f(x)$ can be expanded in a convergent series of positive integral powers of x, between the values of 0 and x, that expansion is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}}f''(0) + \frac{x^3}{\underline{|3}}f'''(0) + \ldots$$

The meaning of $f(0)$, &c., is that $f(x)$ and its successive derivatives (p. 56) are first to be found, and then x must be equated to zero. The two conditions under which Taylor's expansion fails hold also for the failure of Maclaurin's form.

Expand $\sin x$: $f(0) = 0$, $f'(0)$ or $\cos 0 = 1$, $f''(0) = 0$, $f'''(0) = -1$,

so $$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots$$

For ϵ^x, $f(0) = 1$ and every derivative also $= 1$,

so $$\epsilon^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots$$

Lagrange has shown that in Taylor's theorem, if all the derivatives of $f(x)$ up to the nth inclusive are finite and continuous, the remainder R after the h^{n-1} term can be written $\frac{h^n}{\underline{|n}}f^{(n)}(x + \theta h)$, where θ is some proper fraction. If this R cannot be made to vanish in the limit when n is taken sufficiently large, the expansion fails as the series does not approach a finite limit.

The expression for R in Maclaurin's form is $\frac{x^n}{\underline{|n}} f^{(n)}(\theta x)$.

Euler's Theorem.

Euler's theorem deals with partial derivatives.

If u be a homogeneous function of the nth degree, e.g. if u is a series, say $\Sigma A x^a y^b$, where $a + b$ in each term is equal to n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

E.g.

$$u = A_0 x^2 + A_1 xy + A_2 y^2.$$

$$\frac{\partial u}{\partial x} = 2A_0 x + A_1 y.$$

$$\frac{\partial u}{\partial y} = A_1 x + 2A_2 y.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2(A_0 x^2 + A_1 xy + A_2 y^2) = nu.$$

Undetermined Forms.

These are usually but inappropriately called "indeterminate" forms; they comprise such expressions as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, &c.

(1) $\frac{0}{0}$. Suppose that the form is that of a fraction $\frac{\phi(x)}{\psi(x)}$.

Find the derivative of both the numerator and denominator, and form the new fraction $\frac{\phi'(x)}{\psi'(x)}$; if this again gives the same result, repeat the process until a definite result is obtained.

$$\underset{\theta \to 0}{\mathrm{Lt}} \frac{\cos\theta - 1}{\theta^2},\ \frac{-\sin\theta}{2\theta},\ \frac{-\cos\theta}{2} = -\tfrac{1}{2}.$$

(2) The same procedure should be used if $\frac{\phi(x)}{\psi(x)} = \frac{\infty}{\infty}$.

(3) If the form be $0 \times \infty$, it is best to put it in the form $\frac{0}{0}$; suppose $\phi(a) = 0$, $\psi(a) = \infty$.

$$\operatorname*{Lt}_{x \to a} \phi(x)\psi(x) = \frac{\phi(x)}{\dfrac{1}{\psi(x)}}, \text{ and treat as before.}$$

(4) If the form be $\infty - \infty$, say $\phi(x) - \psi(x)$, put it in the form

$$\operatorname*{Lt}_{x \to \infty} \psi(x)\left[\frac{\phi(x)}{\psi(x)} - 1\right].$$

Unless $\dfrac{\phi(x)}{\psi(x)} = 1$, the limiting value must be infinite; if the fraction $= 1$, we have the previous form $\infty \times 0$ (3).

Successive Differentiation.

$y = e^{ax}$. $\qquad \dfrac{d^n y}{dx^n} = a^n e^{ax}$.

$y = a^x = e^{x \lg a}$. $\qquad \dfrac{d^n y}{dx^n} = (\lg a)^n a^x$.

$y = \lg(x + a)$. $\qquad \dfrac{d^n y}{dx^n} = \dfrac{(-1)^{n-1} \underline{|n-1}}{(x + a)^n}$.

$y = \dfrac{1}{x + a}$. $\qquad \dfrac{d^n y}{dx^n} = \dfrac{(-1)^n \underline{|n}}{(x + a)^{n+1}}$.

$y = {}^{\sin}_{\cos} mx$. $\qquad \dfrac{d^{2r} y}{dx^{2r}} = (-m^2)^r \, {}^{\sin}_{\cos} mx$.

$y = {}^{\sin}_{\cos}(ax + b)$. $\qquad \dfrac{d^n y}{dx^n} = a^n \, {}^{\sin}_{\cos}\left(ax + b + \dfrac{n\pi}{2}\right)$.

$y = e^{ax} \, {}^{\sin}_{\cos}(bx + c)$. $\qquad \dfrac{d^n y}{dx^n} = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \, {}^{\sin}_{\cos}\left(bx + c + n \tan^{-1}\dfrac{b}{a}\right)$.

CHAPTER VII

INTEGRATION

Any function whose derivative is the given function in the integrand is called the integral of the integrand;

e.g. $$\int 3x^2\,dx = x^3, \text{ and } \int x^3\,dx = \tfrac{1}{4}x^4.$$

But an arbitrary constant, say C, must always be added to obtain the *general integral*, for a constant term never appears in the derivative. Hence the *general integral* of $\int x^3\,dx$ is $\frac{1}{4}x^4 + C$. When the constant term is neglected we shall merely name the expression obtained as the integral.

It is clear that integration is the reverse of differentiation, so that every unfamiliar integration should be tested by differentiating the integral and so obtaining the integrand.

A great many expressions occur which cannot be mathematically integrated, but for these an approximate integration between limits can often be made by plotting a graph of the curve and calculating its area between these limits. However, there are a few useful devices for reducing the expression to be integrated to a form similar to one of the standard forms given on p. 116–121.

Useful Devices.

(1) *Change of the independent variable.*

If x is a function of u, $\int F(x)dx = \int F(x)\frac{dx}{du}du$; so when the integrand can be more simply integrated in terms of u, replace dx by $\frac{dx}{du}du$.

E.g. $\int x\sqrt{x^2+a}\,dx$. Let $x^2+a=u^2$;

then $$2x\frac{dx}{du}=2u, \quad \text{or} \quad \frac{dx}{du}=\frac{u}{x}.$$

$$\int x\sqrt{x^2+a}\,\frac{dx}{du}\,du=\int u^2du$$

$$=\tfrac{1}{3}u^3=\tfrac{1}{3}(x^2+a)^{\frac{3}{2}}.$$

The whole difficulty lies in the choice of the expression for u; considerable ingenuity is often required in choosing such a value of u as will reduce the original expression to a standard form.

(a) Sometimes in fractional expressions the substitution of $\frac{1}{u}$ for x or for x^n may prove successful, e.g. $\int\frac{dx}{x(a+bx^n)}$.

Let $x^n=u^{-1}$, then

$$nx^{n-1}\frac{dx}{du}=-u^{-2} \quad \text{or} \quad \frac{-x^n}{u};$$

$$\therefore \quad \frac{dx}{du}=\frac{-x^n}{nux^{n-1}}=\frac{-x}{nu}, \quad \text{and} \quad \frac{dx}{x}=-\frac{du}{nu}.$$

We have then $$-\frac{1}{n}\int\frac{du}{u(a+bu^{-1})}=-\frac{1}{n}\int\frac{du}{au+b}$$

$$=-\frac{1}{an}\lg(au+b)=\frac{1}{an}\lg\frac{x^n}{a+bx^n}.$$

(b) Trigonometrical substitutions are often useful.

Thus for $\sqrt{a^2-x^2}$, try $x=a\sin\theta$ or $a\cos\theta$,

for $\sqrt{a^2-(x+b)^2}$, try $x+b=a\sin\theta$ or $a\cos\theta$,

for $\sqrt{a^2+x^2}$, try $x=a\tan\theta$ or $a\cot\theta$,

for $\sqrt{x^2-a^2}$, try $x=a\sec\theta$ or $a\operatorname{cosec}\theta$.

E.g. $\int\sqrt{a^2 - x^2}\,dx$. Let $x = a\sin\theta$;

$$\therefore\ \frac{dx}{d\theta} = a\cos\theta,$$

$$\sqrt{a^2 - x^2}\frac{dx}{d\theta} = a^2\cos^2\theta,$$

$$a^2\int\cos^2\theta\,d\theta = \frac{a^2}{2}\int(1 + \cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}(\theta + \tfrac{1}{2}\sin 2\theta) \text{ or } \frac{a^2}{2}(\sin\theta\cos\theta + \theta)$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

This is an important example.

If $a\cos\theta$ had been substituted for x instead of $a\sin\theta$, the signs would be altered, and the last term would have been $-\frac{a^2}{2}\cos^{-1}\frac{x}{a}$, but clearly the results only differ by a constant, for $\sin^{-1}\frac{x}{a} = \frac{\pi}{2} - \cos^{-1}\frac{x}{a}$, and $\frac{\pi}{2}$ would be included in the arbitrary constant.

(2) *Integration by parts.*

Since $$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

on integrating, we find

$$uv = \int u\,dv + \int v\,du, \text{ or } \int u\,dv = uv - \int v\,du.$$

Hence whenever the integral of v with respect to u, or $\int v\,du$, is known, $\int u\,dv$ can be found. In this case it is simpler and quite allowable to use the differential notation.

(i) E.g. $\int x\cos nx\,dx$. Let $u = x$, $dv = \cos nx\,dx$.

$$du = dx,\ v = \frac{\sin nx}{n}.$$

$$\int x \cos nx\,dx = \frac{x \sin nx}{n} - \int \frac{\sin nx\,dx}{n}$$

$$= \frac{x \sin nx}{n} + \frac{\cos nx}{n^2}.$$

Care must be taken to make the expression $\int v\,du$ as simple for integration as possible. If in this case we take $u = \cos nx$ and dv as $x\,dx$, we obtain a more complex integral than the original one,

$$\frac{x^2}{2} \cos nx + \frac{n}{2} \int x^2 \sin nx\,dx.$$

(ii) $\int \lg x\,dx$. Let $u = \lg x$, $dv = dx$,

$$du = \frac{dx}{x}, \quad v = x.$$

$$\int \lg x\,dx = x \lg x - \int x \frac{dx}{x} = x \lg x - x.$$

This device of substituting u or v for x is often useful. Sometimes the operation may be repeated several times. E.g.

$$\int (\lg x)^2\,dx = x(\lg x)^2 - 2\int \lg x\,dx$$

$$= x(\lg x)^2 - 2x \lg x + 2x.$$

In this way a series for $\int (\lg x)^n\,dx$ can be obtained, and indeed most of the ordinary *reduction formulæ* have been obtained by the method of integration by parts (p. 77).

Integration of Fractions.

All polynomials in x whose coefficients are real can be resolved into real factors of the first and second degrees.

$$F(x) = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots K$$

$$= M(x - \beta)(x - \gamma) \ldots (ax^2 + bx + c) \ldots .$$

Any proper fraction $\dfrac{f(x)}{\phi(x)}$ when $f(x)$ is of lower degree than $\phi(x)$ ($\phi(x)$ having no repeated factors) can be put into partial fractions with real coefficients such as

$$\frac{A}{x-\beta}+\frac{B}{x-\gamma}+\frac{Cx+K}{ax^2+bx+c}+\dots.$$

Each of these terms can be integrated on inspection, or with a little ingenuity can be put into one of the standard forms (see p. 64), e.g.

$$A\int\frac{dx}{x-\beta}=A\lg(x-\beta). \quad A\int\frac{dx}{x^2+a^2}=\frac{A}{a}\tan^{-1}\frac{x}{a}.$$

Note carefully that:

(1) If the numerator of a fraction is the derivative of the square of the denominator, the integral is twice the denominator,

$$\frac{(2ax+b)\,dx}{\sqrt{ax^2+bx+c}}=2\sqrt{(ax^2+bx+c)}.$$

If a factor be required to fulfil this condition, the reciprocal of that factor must be placed before the integral sign. E.g.

$$\int\frac{5x^7\,dx}{x^4}=\tfrac{5}{8}\int\frac{8x^7\,dx}{x^4}=\tfrac{5}{8}(2x^4)=\tfrac{5}{4}x^4.$$

The chief use made of this proceeding is in cases where the denominator is the square root of a function.

(2) If the numerator of a fractional expression be made by a factor equal to the derivative of the denominator, and the reciprocal of the factor be placed before the integral sign, the result on integration will be the logarithm of the denominator. E.g.

$$\int\frac{x^{n-1}dx}{a+bx^n}=\frac{1}{nb}\int\frac{nbx^{n-1}dx}{a+bx^n}=\frac{1}{nb}\lg(a+bx^n).$$

This is the basis on which such expressions as $\displaystyle\int\frac{(px+q)dx}{ax^2+bx+c}$ are integrated.

Partial Fractions.

Let $\dfrac{f(x)}{\phi(x)}$ be a proper fraction when $\phi(x) = (x - \beta)(x - \gamma)$.

$$\frac{f(x)}{\phi(x)} = \frac{f(x)}{(x-\beta)(x-\gamma)} = \frac{A}{(x-\beta)} + \frac{B}{(x-\gamma)},$$

$$f(x) = A(x-\gamma) + B(x-\beta).$$

This is true for all values of x. Let $x = \beta$,

then $\qquad f(\beta) = A(\beta - \gamma) + B(0); \quad \therefore A = \dfrac{f(\beta)}{\beta - \gamma}.$

Again, let $x = \gamma$; then $f(\gamma) = A(0) + B(\gamma - \beta)$.

$$\therefore B = \frac{f(\gamma)}{\gamma - \beta}.$$

In practice a slight modification of this procedure is used, as will be seen in the following example:

$$\frac{8x+5}{(x-2)(x-3)(x+4)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x+4}.$$

To find A multiply throughout by $x - 2$, and then put $x = 2$; the B and C terms will vanish, and we are left with

$$A = \frac{8(2)+5}{(2-3)(2+4)} = \frac{21}{-6} = -\frac{7}{2}.$$

Similarly, to find B, multiply by $x - 3$, and then put $x = 3$.

$$B = \frac{8(3)+5}{(3-2)(3+4)} = \frac{29}{7}$$

and $\qquad C = \dfrac{8(-4)+5}{(-4-2)(-4-3)} = -\dfrac{27}{42} = -\dfrac{9}{14}.$

$$\therefore \int \frac{(8x+5)\,dx}{(x-2)(x-3)(x+4)}$$

$$= -\tfrac{7}{2}\lg(x-2) + \tfrac{29}{7}\lg(x-3) - \tfrac{9}{14}\lg(x+4).$$

(i) *Case of Equal Roots.*—If one of the factors in the denominator be $(x-\gamma)^r$, there will be a sequence of r partial fractions, e.g.

$$\frac{A_1}{(x-\gamma)}, \frac{A_2}{(x-\gamma)^2}, \ldots \frac{A_r}{(x-\gamma)^r}.$$

In such a case the procedure must be modified as below, or in some other way:

$$\frac{8x+5}{(x-2)^3(x+4)} = \frac{A}{(x-2)^3} + \frac{B}{(x-2)^2} + \frac{C}{x-2} + \frac{D}{x+4}.$$

We evaluate A and D in the way described above:

$$A = \frac{8(2)+5}{2+4} = \frac{21}{6} = \frac{7}{2}. \quad D = \frac{8(-4)+5}{(-4-2)^3} = \frac{-27}{-216} = \frac{1}{8}.$$

The values of B and C are found by equating the coefficients of powers of x on either side of the equation:

$$8x+5 = A(x+4) + B(x^2+2x-8)$$
$$+ C(x^3-12x+16) + D(x^3-6x^2+12x-8).$$

There are no cubes and no squares of x on the left side. So

$$0 = C + D, \therefore C = -D = -\tfrac{1}{8};$$
$$0 = B - 6D, \therefore B = 6D = \tfrac{6}{8} = \tfrac{3}{4}.$$

So $$\frac{8x+5}{(x-2)^3(x-4)} = \frac{7}{2(x-2)^3}$$
$$+ \frac{3}{4(x-2)^2} - \frac{1}{8(x-2)} + \frac{1}{8(x+4)},$$

and $$\int \frac{(8x+5)dx}{(x-2)^3(x+4)} = \frac{-7}{4(x-2)^2} - \frac{3}{4(x-2)}$$
$$- \tfrac{1}{8}\lg(x-2) + \tfrac{1}{8}\lg(x+4).$$

(ii) *Quadratic Factors.*—In many cases the complex

algebraic function cannot be resolved into real factors of the first degree, but whenever the coefficients are real it can be resolved into factors of the second degree with real coefficients. The partial fractions are of the type

$$\frac{A_1x+B_1}{ax^2+bx+c}, \quad \frac{A_2x+B_2}{(ax^2+bx+c)^2},$$

and so on; only the first of these being used for a non-repeated factor.

Integration of $\displaystyle\int\frac{(px+q)\,dx}{ax^2+bx+c}$.

We have seen that when $b^2>4ac$, i.e. when the denominator can be expressed by $a(x-\beta)(x-\gamma)$, the method of integration by partial fractions should be used.

Let ax^2+bx+c be denoted by $\phi(x)$, and its first derivative by $\phi'(x)$. Now as we can integrate with regard to x both $\dfrac{\phi'(x)}{\phi(x)}$ and $\dfrac{\phi'(x)}{\sqrt{\phi(x)}}$, in the next two sections we shall replace $px+q$ by $\lambda\phi'(x)+\mu$, i.e. by $\lambda(2ax+b)+\mu$, and after equating coefficients of x on both sides, we obtain

$$\lambda=\frac{p}{2a} \quad\text{and}\quad \mu=q-b\lambda=\frac{2aq-bp}{2a}.$$

(1) When $b^2=4ac$,

$$\phi(x)=ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2, \quad\text{and}$$

$$\int\frac{px+q}{ax^2+bx+c}\,dx=\lambda\int\frac{2ax+b}{ax^2+bx+c}\,dx+\frac{\mu}{a}\int\frac{dx}{\left(x+\dfrac{b}{2a}\right)^2}$$

$$=\frac{p}{2a}\lg(ax^2+bx+c)-\frac{2aq-bp}{2a^2\left(x+\dfrac{b}{2a}\right)}$$

or $$=\frac{p}{2a}\lg(ax^2+bx+c)-\frac{2aq-bp}{a(2ax+b)}.$$

(1′) When $p = 0$, and $c = \dfrac{b^2}{4a}$,

$$\int \frac{q\,dx}{ax^2 + bx + c} = -\frac{2aq}{a(2ax + b)} \text{ or } \frac{-q}{ax + \frac{1}{2}b}.$$

(2) When $b^2 < 4ac$,

$$ax^2 + bx + c = a\left\{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right\}.$$

Denote $\left(\dfrac{2ax + b}{2a}\right)^2$ by X^2, and $\left(\dfrac{4ac - b^2}{4a^2}\right)$ by P^2.

Then when $px + q = \lambda(2ax + b) + \mu$,

$$\int \frac{(px + q)\,dx}{ax^2 + bx + c} = \lambda \int \frac{(2ax + b)\,dx}{ax^2 + bx + c} + \frac{\mu}{a} \int \frac{dx}{X^2 + P^2}$$

$$= \frac{p}{2a} \lg(ax^2 + bx + c) + \frac{\mu}{aP} \tan^{-1} \frac{X}{P},$$

where $\dfrac{\mu}{aP} \tan^{-1} \dfrac{X}{P} = \dfrac{(2aq - bp)2a}{2a^2\sqrt{4ac - b^2}} \tan^{-1} \dfrac{2ax + b}{\sqrt{4ac - b^2}}.$

Hence $\displaystyle\int \frac{(px + q)\,dx}{ax^2 + bx + c}$

$$= \frac{p}{2a} \lg(ax^2 + bx + c) + \frac{2aq - bp}{a\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}}.$$

Integration of $\displaystyle\int \frac{(px + q)\,dx}{\sqrt{ax^2 + bx + c}}$.

By using the same values for λ and μ, the integral is

$$\lambda \int \frac{(2ax + b)\,dx}{\sqrt{ax^2 + bx + c}} + \mu \int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$

The first expression is obviously $2\lambda\sqrt{ax^2 + bx + c}$. The second expression requires a different treatment according as b^2 is greater or less than $4ac$.

(1) When $b^2 > 4ac$,

$$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)}.$$

On writing X^2 for $\left(\frac{2ax+b}{2a}\right)^2$ and $P_1^{\,2}$ for $\left(\frac{b^2-4ac}{4a^2}\right)$,

$$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{X^2-P_1^{\,2}},$$

$$\mu\int\frac{dx}{\sqrt{ax^2+bx+c}} = \frac{\mu}{\sqrt{a}}\int\frac{dx}{\sqrt{X^2-P_1^{\,2}}}$$

$$= \frac{2aq-bp}{2a\sqrt{a}}\cosh^{-1}\frac{X}{P_1},$$

or $$\frac{2aq-bp}{2a\sqrt{a}}\lg\frac{2ax+b+2\sqrt{a^2x^2+abx+ac}}{\sqrt{b^2-4ac}}.$$

(2) When $b^2 < 4ac$,

$$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c}{a}-\frac{b^2}{4a^2}\right)},$$

$$\text{or}\quad \sqrt{a}\sqrt{X^2+P^2},$$

and $$\frac{\mu}{\sqrt{a}}\int\frac{dx}{\sqrt{X^2+P^2}} = \frac{2aq-bp}{2a\sqrt{a}}\sinh^{-1}\frac{X}{P},$$

or $$\frac{2aq-bp}{2a\sqrt{a}}\lg\frac{2ax+b+2\sqrt{a^2x^2+abx+ac}}{\sqrt{4ac-b^2}}.$$

In each case, (1) and (2), the previous term

$$2\lambda\sqrt{ax^2+bx+c} \text{ or } \frac{p}{a}\sqrt{ax^2+bx+c}$$

must be added.

Tables of hyperbolic functions are not usually available, but the logarithmic result is always applicable. Find the ordinary logarithm to base 10, and multiply it by m or $\lg 10$ (i.e. 2·302585) (p. 112) to obtain $\lg n$.

Integration of $\int \frac{(px + q)\,dx}{\sqrt{c + bx - ax^2}}$.

This is real when $c + bx > ax^2$, and is similarly found to be

$$-\frac{p}{a}\sqrt{c + bx - ax^2} + \frac{2aq + bp}{2a\sqrt{a}} \sin^{-1} \frac{2ax - b}{\sqrt{4ac + b^2}}.$$

Series.

The integrand may often be expressed in the form of a convergent power series; if this be the case, the integral will be the sum of the integrals of each term.

Approximate Integration. *Simpson's Rule.*

Suppose that we wish to find the area of the cross-section of a ship. We know and can trace the contour of this cross-section, but we do not know the equation of the curve. We wish to find $\int f(x)dx$, but we do not know $f(x)$, although for any value of x we know approximately the value of y. A similar problem arises frequently in a laboratory when a table is made of numerical measurements after the lapse of equal intervals of time. Or we may know $f(x)$ but we cannot integrate it.

In these cases if we wish to find the value of $\int_a^b f(x)dx$ or the area between the curve, the axis of x, and the ordinates $x = a$ and $x = b$, we draw a horizontal abscissa $b - a$ in length, and divide the abscissa into an *even* number of equal intervals (of length h). Then erect ordinates (y) at each of these marked divisions to indicate the value of y when $x = a, a + h, a + 2h, \ldots b$.

Simpson's rule is: "Add together the two extreme ordinates and call the sum A, add together the *remaining odd* ordinates and call the sum B, add together the *even* ordinates and call the sum C; then the area or

$$\int_a^b f(x)dx \approx \frac{h}{3}(A + 2B + 4C).$$

This is practically dividing the sinuous curve into a number of consecutive small parabolic arcs. The number of ordinates must be odd, as the number of intervals must be even, and the greater the number, the more accurate will be the result. Seven ordinates are sufficient for most purposes, but more should be taken if the contour does not resemble a succession of parabolic arcs.

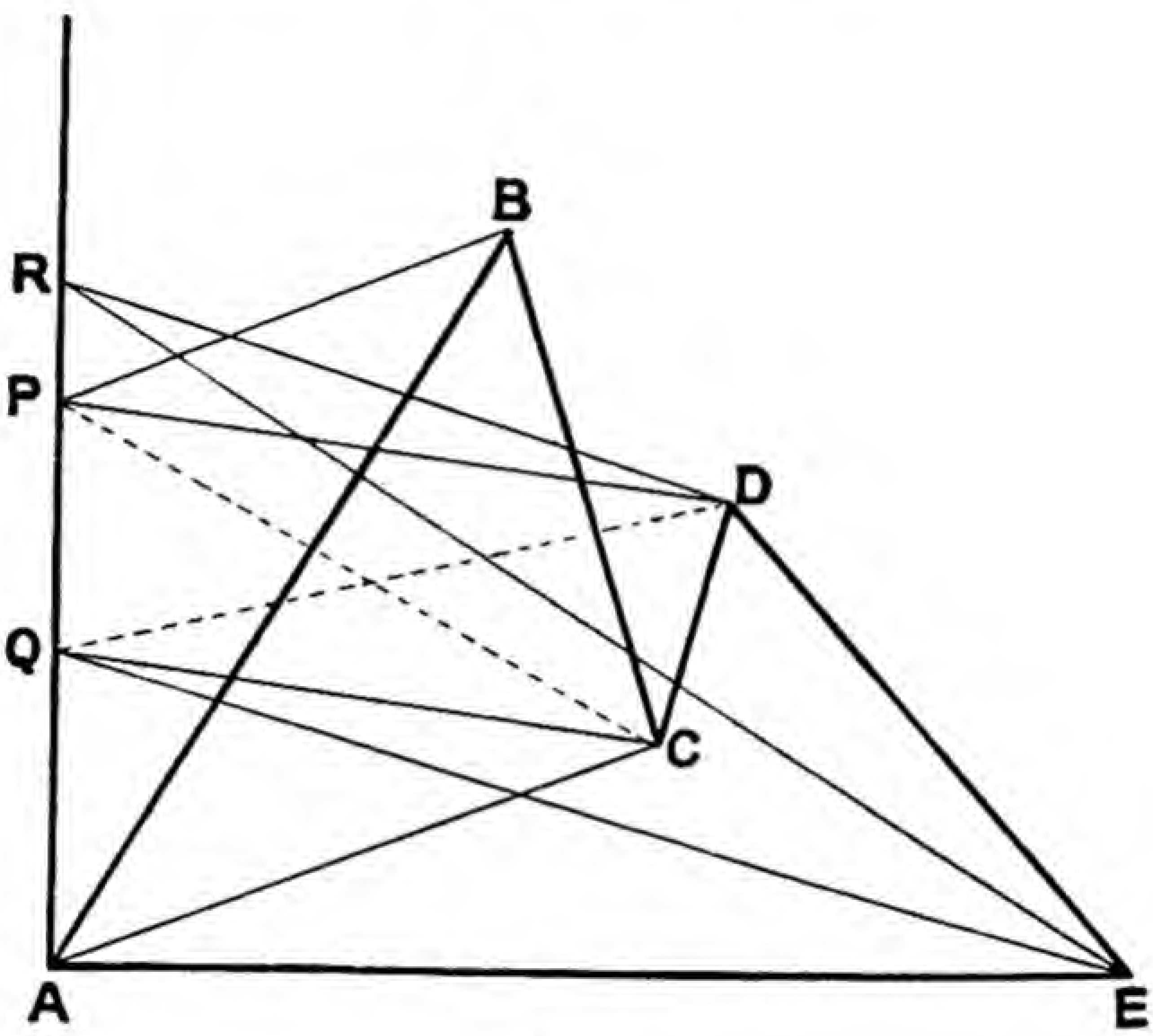

Fig. 4

Simple Rapid Device for rough approximation.

Suppose that we require the area of $ABCDE$ (fig. 4). Let AE represent the axis of x, and erect a perpendicular at A as axis of y; then $ABCDE = \int y\,dx$ or $\int_0^b f(x)\,dx$ between the limits $x = AE$ or b and $x = 0$. If it be a laboratory experiment, and observations are made at noted intervals of time, AE will denote the time of the last observation, when A denotes zero time. The area $ABCDE$ will then denote $\int_0^b f(t)\,dt$. In this case there is a re-entrant angle at C, but it will be found that this makes no difference to the procedure, which is perfectly general.

Draw AC to the second point of the curve beyond A,

and through B (the preceding point) draw PB parallel to AC, meeting the OY axis in P. Now draw PD to the second point beyond B, and then draw through C the line QC parallel to PD. Draw QE to the second point beyond C, and RD parallel to QE, and so on, if necessary. But as E is the last point of the figure, join RE. The area $ABCDE$ is equal to that of the right-angled triangle ARE or $\frac{1}{2}AR \times AE$.

$$PB \parallel AC; \quad \therefore\ ACB = ACP \text{ or } APC. \qquad (1)$$
$$QC \parallel PD; \quad \therefore\ PDC = PDQ \text{ or } QPD. \qquad (2)$$
$$RD \parallel QE; \quad \therefore\ QED = QER. \qquad (3)$$

$$\begin{aligned} ABC + ACDE &= APC + ACDE \text{ by } (1) \\ &= APDE - PDC = APDE - QPD \text{ by } (2) \\ &= AQE + QED = AQE + QER \text{ by } (3) \\ &= ARE. \end{aligned}$$

If the first side of the polygon be at right angles to the last side, e.g. $AQCDE$, AQ will be on the y axis, and QD will be drawn to the second point of the curve not on the y axis, and RC will be drawn parallel to QD. Then as E is the second point beyond C, join RE, and SD will be drawn parallel to RE; finally join SE. The reader is expected to make this diagram for himself.

$$RC \parallel QD; \quad \therefore\ QDC = QDR. \qquad (1)$$
$$SD \parallel RE; \quad \therefore\ RED = RES. \qquad (2)$$

$$\begin{aligned} AQCDE &= AQDE - QDR \text{ by } (1) \\ &= ARDE = ARE + RES \text{ by } (2) \\ &= AES. \end{aligned}$$

It will be noted that none of these parallel lines need really be drawn, except for the proof. All that is necessary is to find the points P, Q, R, &c., on the axis OY, i.e. mark P where PB would be parallel to AC, Q where QC would be parallel to PD, and so on. It can be done very rapidly with a sector, keeping one leg of the sector always in the OY line, the other leg opened out to coincide with AC, then slid upwards until the leg coincides with PB, then inclined to PD, and then slid upwards until it coin-

cides with QC, and so on. Note that when using this method there is no need to make the intervals equal.

Many cases will occur when approximate integration between definite limits is required (e.g. indicator diagrams). There should be no error with this rapid method if the area be that of a rectilinear polygon, but no general estimate of the error can be given if the bounding line be a curve. The error of Simpson's method will rarely exceed 2 per cent.

Occasionally we may be given $P = k\int_0^b \phi(t)\,dt$, but we wish to determine the constant k. If we can determine S or $\int_0^b \phi(t)\,dt$ as above or by Simpson's method, $k = \frac{P}{S}$.

Definite Integrals.

If we wish to find the area between a curve and the axis of x between two ordinates at the points $x = a$ and $x = b$, the expression is written $A = \int_a^b y\,dx$, and it is read "the integral from a to b of $y\,dx$"; a is called the lower limit and b the upper limit of the integral. In this case the word limit has merely the meaning of one of the "end-values" of the variable. Find the integral of the expression, replace x by b, then replace x by a, and subtract the second result from the first. Note that owing to the subtraction the arbitrary constant always disappears in definite integrals.

To find the area of the quadrant of a circle of radius r.

$A = \int_0^r y\,dx$. Here $y = r\sin\theta$, $x = r\cos\theta$, $dx = -r\sin\theta\,d\theta$, and when $x = 0$, $\theta = \frac{\pi}{2}$; when $x = r$, $\theta = 0$.

$$A = -r^2\int_{\frac{\pi}{2}}^0 \sin^2\theta\,d\theta = -\frac{r^2}{2}\int_{\frac{\pi}{2}}^0 (1-\cos 2\theta)\,d\theta$$

$$= -\frac{r^2}{2}\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_{\frac{\pi}{2}}^0 = \frac{\pi r^2}{4}.$$

In the transformation the lower limit is greater than the

upper limit; pay no attention to this, which is due to the fact that as x increases θ decreases. In every case subtract the so-called lower limit from the upper, and the remainder will give the required result.

We are here really dealing with our previous (p. 59) integral $\int\sqrt{a^2 - x^2}dx$, but we are now taking $x = r\cos\theta$ instead of $a\sin\theta$ as before. We shall find, however, that the limits are different, and the evaluation of the definite integral for the quadrant of a circle under these conditions is exactly the same.

If $x = a\sin\theta$; $y = a\cos\theta$; when $x = 0$, $\theta = 0$; when $x = a$, $\theta = \frac{1}{2}\pi$; and $dx = a\cos\theta\, d\theta$:

$$A = a^2\int_0^{\frac{1}{2}\pi}\cos^2\theta\, d\theta = \frac{a^2}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\frac{1}{2}\pi} = \frac{\pi a^2}{4}.$$

Mean Values.

The mean value of a continuous function over any range from a to b is equal to some value of the independent variable within that range. If there be n equidistant values (h) of x throughout the range a to b, the limiting value to which the arithmetic mean $\frac{1}{n}(y_1 + y_2 + \ldots + y_n)$ tends as n is indefinitely increased and h diminished is called the mean value of the function over the range a to b. As $h = \frac{b-a}{n}$, the previous expression becomes

$$\frac{y_1h + y_2h + \ldots y_nh}{b-a} \quad \text{or} \quad \int_a^b\frac{F(x)dx}{b-a}.$$

The mean value of $\sin\theta$ for equidistant intervals of θ ranging from 0 to π is

$$\frac{1}{\pi}\int_0^{\pi}\sin\theta d\theta = \frac{1}{\pi}\Big[-\cos\theta\Big]_0^{\pi} = \frac{2}{\pi}.$$

The mean value of $\sin\theta$ is then $\cdot 6366$, and of the ordinate $\cdot 6366r$. But if the ordinates were drawn through equidis-

tant points on the *diameter*, a different result would be obtained, so it is essential to have a clear understanding as to what is the independent variable.

$$\frac{1}{2r}\int_{-r}^{r} y\,dx = -\frac{r}{2}\int_{-\pi}^{0} \sin^2\theta\,d\theta \text{ (p. 70)}$$

$$= -\frac{r}{4}\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_{\pi}^{0} = \frac{\pi r}{4} = \cdot 7854r.$$

An alternating electric current is represented by the graph of $y = a\sin nt$; the mean value of the ordinate for a half period $\left(\frac{\pi}{n}\right)$ from $t = 0$ to $t = \frac{\pi}{n}$, is

$$\frac{n}{\pi}\int_{0}^{\frac{\pi}{n}} a\sin nt\,dt = \frac{2}{\pi}a \quad\text{or}\quad \cdot 6366a.$$

The square of the ordinate is y^2 or $a^2\sin^2 nt$. The mean value of the square of the ordinate is

$$\frac{n}{\pi}\int_{0}^{\frac{\pi}{n}} a^2\sin^2 nt\,dt = \frac{na^2}{2\pi}\Big[t - \frac{1}{2n}\sin 2nt\Big]_{0}^{\frac{\pi}{n}} = \frac{a^2}{2}.$$

The square root of this mean value is $\frac{a}{\sqrt{2}}$, and is called by electricians the R.M.S. (root mean square) of the ordinate, and is of great importance in alternating current theory.

Alteration of Limits.

Assuming that the limits of integration (e.g. a to b) are finite, and that y or $\phi(x)$ is finite throughout the entire range considered, we may make the following statements.

(1) A definite integral is a function of its limits, not of its variable.

If $$D\psi(x) = \phi(x),$$

$$\int_a^b \phi(x)\,dx = \int_a^b \phi(z)\,dz = \psi(b) - \psi(a).$$

Hence, if to effect an integration a transformation from

x to u, say, is required, it is quite unnecessary to effect the troublesome change back from u to x; one need only express the limits in terms of the u variable, as we have done above.

(2) The limits of an integral can be interchanged if at the same time the sign of the integral is changed.

$$\int_a^b \phi(x)dx = -\int_b^a \phi(x)dx.$$

The evaluation of definite integrals is greatly simplified if we distinguish between *even* and *odd* functions.

Even functions are those in which $\phi(-x) = \phi(x)$.

E.g. x^{2n}, $\cos^n x$, $\cosh^n u$, their reciprocals and any even powers of odd functions.

Odd functions are those in which $\phi(-x) = -\phi(x)$.

E.g. x^{2n-1}, $\sin^{2n-1}x$, $\tan^{2n-1}x$, $\sinh^{2n-1}u$, $\tanh^{2n-1}u$ and their reciprocals.

(3) If $\phi(x)$ be an *even* function, $\int_{-b}^b \phi(x)dx = 2\int_0^b \phi(x)dx.$

Thus
$$\int_{-b}^b x^2 dx = 2\int_0^b x^2 dx = 2\frac{b^3}{3}.$$

$$\int_{-a}^a \cos x\,dx = 2\int_0^a \cos x\,dx = 2\sin a.$$

(3′) If $\phi(x)$ be an *odd* function, $\int_{-b}^b \phi(x)dx = 0.$

Thus
$$\int_{-b}^b x\,dx = 0;\quad \int_{-a}^a \sin x\,dx = 0.$$

The reason, of course, is that on integration even functions become odd and vice versa.

(4)
$$\int_0^b \phi(x)dx = \int_0^b \phi(b-x)dx,$$

for $\int\phi(b-x)dx = -\psi(b-x)$, as dx is negative.

E.g. $$\int_0^b x\,dx = \int_0^b (b-x)\,dx = \frac{b^2}{2}.$$

(4′) If $\phi(b-x)=\phi(x)$, $\int_0^b \phi(x)\,dx = 2\int_0^{\frac{1}{2}b} \phi(x)\,dx.$

E.g. $$\int_0^{\pi} \sin x\,dx = 2\int_0^{\frac{1}{2}\pi} \sin x\,dx = 2.$$

(4″) If $\phi(b-x) = -\phi(x)$, $\int_0^b \phi(x)\,dx = 0.$

E.g. $$\int_0^{\pi} \cos x\,dx = 0.$$

(5) An especially important case is

$$\int_0^{\frac{1}{2}\pi} \sin x\,dx = \int_0^{\frac{1}{2}\pi} \cos x\,dx:$$

$$\int_0^{\frac{1}{2}\pi} \sin x\,dx = \int_0^{\frac{1}{2}\pi} \sin(\tfrac{1}{2}\pi - x)\,dx = \int_0^{\frac{1}{2}\pi} \cos x\,dx.$$

A useful example is

$$\int_0^{\frac{1}{2}\pi} \sin^2 x\,dx = \int_0^{\frac{1}{2}\pi} \cos^2 x\,dx$$
$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (\cos^2 x + \sin^2 x)\,dx = \tfrac{1}{4}\pi.$$

Generally, $$\int_0^{\frac{1}{2}\pi} \phi(\sin x)\,dx = \int_0^{\frac{1}{2}\pi} \phi(\cos x)\,dx.$$

If $\int_0^{\pi} (1 + 4\cos\theta + 5\cos^2\theta + 2\cos^3\theta)\,d\theta$ is to be integrated, odd powers of $\cos\theta$ will vanish by (4″), and we have only to integrate

$$\int_0^{\pi} (1 + 5\cos^2\theta)\,d\theta \text{ or } 2\int_0^{\frac{1}{2}\pi} (1 + 5\cos^2\theta)\,d\theta$$
$$= 2\left\{\frac{\pi}{2} + 5\left(\frac{1}{2}\,\frac{\pi}{2}\right)\right\} = 3{\cdot}5\pi.$$

If the function is a trigonometrical function of sine or cosine, it is advantageous to alter the limits when possible

to $\frac{1}{2}\pi$ and 0, as the integration may then be written down at once. An expression is given on p. 77 for the evaluation of the integrals of $\sin^n x dx$ and $\sin^n x \cos^m x dx$ between the limits of 0 and $\frac{1}{2}\pi$, whether the values of m and n be odd or even positive integers. These integrals may also be calculated by *gamma functions* (pp. 75–78, 122–125), as also may other integrals that turn up in practical work.

When $\phi(x)$ becomes Infinite.

If $\phi(x)$ becomes infinite at or between the limits of integration, the range $(b - a)$ must be broken up into intervals which have no infinite values. We will deal with cases in which the upper limit becomes infinite.

Consider $\int_0^1 \frac{dx}{\sqrt{1-x}}$. $\frac{1}{\sqrt{1-x}}$ becomes infinite if $x = 1$. Let the upper limit be reduced by a value e where e has a small positive value.

$$\int_0^{1-e} \frac{dx}{\sqrt{1-x}} = \Big[-2\sqrt{1-x}\Big]_0^{1-e} = 2 - 2\sqrt{e},$$

and as e is indefinitely diminished this tends to the limit 2, and 2 is taken as the value of the definite integral.

If no finite limit is attained when e is indefinitely diminished, the expression cannot be integrated.

This method may also be used if the lower limit gives an infinite value to the function.

Gamma Functions.

The definite integral $\int_0^\infty e^{-x}x^{n-1}dx = \Gamma(n)$.

When n is a positive integer $\Gamma(n+1) = \underline{|n}$.

In every case $\Gamma(n+1) = n\Gamma(n)$.

$\Gamma(2) = \Gamma(1) = 1$; between $\Gamma(2)$ and $\Gamma(1)$ the values are fractional, for which tables of their logarithms are given (p. 122).

$\Gamma(\frac{1}{2}) = \sqrt{\pi} = 1{\cdot}77245385$ and $\log\sqrt{\pi} = {\cdot}2485749$.

$\Gamma(0) = \infty$. $\Gamma(-n)$ is infinite if n be integral. For all values of n, $\Gamma(n)\Gamma(1-n)\sin n\pi = \pi$.

If n be a positive fraction $\Gamma(-n)$ is always finite. When $-n$ is a negative improper fraction, $\Gamma(-n)$ is positive if the integral part of the mixed number be *odd*, but negative if it be *even*. Thus

$$\Gamma(-\cdot 5) = -2\sqrt{\pi};\ \Gamma(-2\cdot 5) = -\tfrac{8}{15}\sqrt{\pi};$$

but

$$\Gamma(-1\cdot 5) = \tfrac{4}{3}\sqrt{\pi}.$$

The gamma function of any number with not more than three decimal places can be evaluated from the tables (p. 122) with the help of this formula:

$$\Gamma(p+n) = (p+n-1)(p+n-2)\ldots p\Gamma(p).$$

If p be a proper or a negative fraction, $p+n$ can be made equal to one of the tabular numbers; if the number be a positive mixed number above 2, it is written as $p+n$ and p is taken as one of the tabular numbers.

By form (1) on the next page a definite integral between the limits $\frac{\pi}{2}$ and 0 whose integrand is a root of a trigonometric function can be easily evaluated.

For instance, $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\,d\theta$ may be written

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}-1}\theta\cos^{1-1}\theta\,d\theta = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{2\Gamma\frac{5}{4}}.$$

Now $\Gamma\ 1\cdot 75 = \frac{3}{4}\,\Gamma\frac{3}{4}$, so $\Gamma\frac{3}{4} = \frac{4}{3}\Gamma\ 1\cdot 75$.

$$\frac{2\Gamma 1\cdot 75\sqrt{\pi}}{3\Gamma 1\cdot 25} = 1\cdot 19814.$$

		log 2	$\cdot 30103$
log $\Gamma 1\cdot 25$	$\bar{1}\cdot 957321$	log $\Gamma\ 1\cdot 75$	$\bar{1}\cdot 963345$
log 3	$\cdot 477121$	log $\sqrt{\pi}$	$\cdot 248575$
	$\cdot 434442$		$\cdot 512950$
			$\cdot 434442$
		log $1\cdot 19814 =$	$\cdot 078508$

This is enough to enable the reader to make an intelligent use of gamma functions, as they usually occur in a very elementary form in practical laboratory work.

Some Definite Integrals.

$$\int_0^{\frac{\pi}{2}} \sin^{2n}\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^{2n}\theta d\theta$$

$$= \frac{(2n-1)(2n-3)\ldots(3)(1)}{(2n)(2n-2)\ldots(4)(2)}\,\frac{\pi}{2}.$$

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta d\theta$$

$$= \frac{(2n-2)(2n-4)\ldots(4)(2)}{(2n-1)(2n-3)\ldots(5)(3)}.$$

If the index of the power be *even*, the series of factors terminates with $\frac{1}{2}\,\frac{\pi}{2}$; but if the index be odd the last factor is $\frac{2}{3}$.

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta d\theta$$

$$= \frac{(m-1)(m-3)\ldots\times(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots}\times f,$$

where f is $\frac{\pi}{2}$ if the positive integers m and n are *both* even, but otherwise $f=1$. Each of the three series of factors is to be continued so long as the factors are positive. All these rules are summed up in one expression by *Gamma Functions*. Let the index of the sine function be $p-1$ and of the cosine function be $q-1$; (if for instance the cosine function be absent $q=1$); then

$$(1)\quad \int_0^{\frac{\pi}{2}} \sin^{p-1}\theta \cos^{q-1}\theta d\theta = \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}{2\Gamma\left(\frac{p+q}{2}\right)}.$$

(2) $\int_0^1 x^{n-1}(1-x)^{m-1}dx = \int_0^\infty \frac{y^{n-1}dy}{(1+y)^{n+m}} = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)}.$

(i) We may verify the value of the first integral in (2) when m and n are positive integers: on integrating by parts

$$\int x^{n-1}(1-x)^{m-1}\,dx = \frac{x^n}{n}(1-x)^{m-1} + \frac{m-1}{n}\int x^n(1-x)^{m-2}dx.$$

Between the limits 0 and 1, $\frac{x^n}{n}(1-x)^{m-1}$ vanishes, and on repeating the process we obtain

$$I = \int_0^1 x^{n-1}(1-x)^{m-1}dx = \frac{(m-1)(m-2)\ldots 1}{n(n+1)(n+2)..(n+m-1)}$$

$$= \frac{\underline{|m-1}\,\underline{|n-1}}{\underline{|m+n-1}} = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)}.$$

(ii) In (2) let $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$; when $x=1$, $\theta = \frac{1}{2}\pi$, &c.

So $\int_0^{\frac{1}{2}\pi} \sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta = \frac{\Gamma n\,\Gamma m}{2\Gamma(n+m)} = \frac{\Gamma\frac{1}{2}p\,\Gamma\frac{1}{2}q}{2\Gamma\frac{1}{2}(p+q)},$

if $p = 2n$, $q = 2m$. This deduces (1) from (2).

(iii) Let $x = \frac{y}{1+y}$ or $1-x = \frac{1}{1+y}$, $-dx = -\frac{dy}{(1+y)^2}.$

$$I = \int_0^\infty \frac{y^{n-1}dy}{(1+y)^{n+m}} = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)}.$$

This proves the equality of the two integrals in (2).

(3) $\int_0^\infty e^{-a^2x^2}dx \quad = \int_0^\infty \frac{e^{-y}dy}{2a\sqrt{y}} = \frac{1}{2a}\Gamma(\tfrac{1}{2}) = \frac{1}{2a}\sqrt{\pi}.$

(4) $\int_0^1 x^m \lg\left(\frac{1}{x}\right)^n dx \quad = \frac{\Gamma(n+1)}{(m+1)^{n+1}}.$

	Cartesians	Polars
Rectification.	$s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx, \quad y=f(x).$ $s=\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy, \quad x=f(y).$ $s=\int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt, \quad x=f(t).\ y=\varphi(t).$	$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta, \quad r=f(\theta).$ $s=\int\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}dr, \ \theta=f(r).$
Area between the curve and	Two given ordinates, $(x=a,\ x=b)$, $A=\int_b^a y\,dx$, and the axis of x. Two given abscissæ, $(y=c,\ y=d)$, $A=\int_d^c x\,dy$, and the axis of y.	Two radii vectores, $(\theta=\alpha,\ \theta=\beta)$, $A=\frac{1}{2}\int_\beta^\alpha r^2 d\theta.$
Volume of revolution about the	x axis, $V=\pi\int y^2dx.$ y axis, $V=\pi\int x^2dy.$	Initial line, $V=\frac{2}{3}\pi\int r^3\sin\theta\,d\theta.$
Surface of revolution about the	x axis, $S=2\pi\int y\,ds.$ y axis, $S=2\pi\int x\,ds.$	Initial line, $S=2\pi\int r\sin\theta\frac{ds}{d\theta}d\theta.$

	Cartesians	Polars
Centroids.	$\bar{x} = \dfrac{\int\int \sigma x\, dx\, dy}{\int\int \sigma\, dx\, dy}$. $\bar{y} = \dfrac{\int\int \sigma y\, dx\, dy}{\int\int \sigma\, dx\, dy}$.	$\bar{x} = \dfrac{\int\int r \cos\theta\, \sigma r\, d\theta\, dr}{\int\int \sigma r\, d\theta\, dr}$. $\bar{y} = \dfrac{\int\int r \sin\theta\, \sigma r\, d\theta\, dr}{\int\int \sigma r\, d\theta\, dr}$.
Radius of gyration (k) about its axis of a uniform solid of revolution.	$k^2 = \dfrac{\int y^4\, dx}{2\int y^2\, dx}$.	$k^2 = \dfrac{\int r^4 \sin^4\theta\, d(r\cos\theta)}{2\int r^2 \sin^2\theta\, d(r\cos\theta)}$.

Dr. Routh's rule.

$$\begin{array}{c}\text{Moment of inertia about}\\ \text{a principal axis}\end{array} = \text{mass}\, \frac{\begin{array}{c}\text{(sum of squares of}\\ \text{perpendicular semi-axes)}\end{array}}{3,\ 4,\ \text{or}\ 5}.$$

The denominator is to be 3, 4, or 5 according as the body is rectangular, elliptical or ellipsoidal.

CHAPTER VIII

Differential Equations

Order.

The order of a differential equation is determined by that of the *highest derivative* that occurs in it.

Thus $\left(\dfrac{d^2y}{dx^2}\right) - x^3\left(\dfrac{dy}{dx}\right)^3 = 0$ is of the *second order.*

Degree.

The degree is the power to which that highest derivative is raised when the equation is in rational form.

Thus the equations $\left(\dfrac{d^3y}{dx^3}\right)^2 + \left(x\dfrac{dy}{dx}\right)^3 = x^4$, and

$$\left(1 + \frac{dy}{dx}\right)^{\frac{3}{2}} = x^2\frac{d^3y}{dx^3},$$

or, in rational form, $\left(1 + \dfrac{dy}{dx}\right)^3 = x^4\left(\dfrac{d^3y}{dx^3}\right)^2$,

are both of the second degree and of the third order.

Linear.

A linear equation is one that involves the dependent variable (y) and its derivatives in the first degree only without products; e.g.

$$(1 + x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x,$$

but an equation that involves a term $y\dfrac{dy}{dx}$ is not linear.

Linear Equations with Constant Coefficients.

As many of the equations that arise in engineering practice and in the practical work of a laboratory are of linear form with coefficients that are regarded as constant, or those that can be reduced to this form, and as these admit of a ready solution by the symbolic method, we shall at first confine our attention to them.

Very often the experimental results agree so closely with the assumption made that the differential equation employed is hastily regarded as expressing a law of the universe. Newton's law of cooling may be taken as an example, viz. that rate of cooling is proportional to the difference of temperature (θ) between the hot body and that of its surroundings, or

$$\frac{d\theta}{dt} = -a\theta; \quad \therefore \int \frac{d\theta}{\theta} = -a\int dt \text{ or } \lg\theta = -at + C.$$

Practically it is found that Newton's law corresponds with observations only when the difference of temperature does not exceed 36° Fahr. Probably Stefan's law of cooling is the correct one which involves the fourth power of the absolute temperature.

If the instrument used will only give a reading of three figures, small variations will be passed unnoticed; indeed experiments must be repeated again and again for different ranges before any approximately reliable law can be attained.

Differential equations are all-important in laboratory research work, but they should be used as first approximation trials to be tested again by further experiments.

A great deal of time and trouble is saved by a little intelligent guessing, provided that each guess is checked by differentiation and seeing that the original equation is obtained thereby. It will be found that although the subject of Differential Equations is notoriously difficult, when we confine ourselves to Linear Equations with constant coefficients, the solution is usually very easy.

Let us take for example the simplest possible equation of the third order, $\frac{d^3y}{dt^3} = 0$.

The first step is obviously $\frac{d^2y}{dt^2} = G$, where G is any arbitrary constant. In the following pages arbitrary constants in the solution will always be denoted by capital letters.

Now whenever all the y's and dy's can be put on one side of the equation, and all the t's and dt's on the other side, the solution is simply obtained by successive integration, whenever the other side has a significant meaning.

$$\int d^2y = \int G\,dt\,dt; \quad \therefore dy = (Gt + U)dt.$$

$$\int dy = \int (Gt + U)\,dt; \quad \therefore\ y = \tfrac{1}{2}Gt^2 + Ut + C.$$

From the differential equation of the third order $\left(\frac{d^3y}{dt^3} = 0\right)$ we have then three arbitrary constants, G, U, and C, and it will always be found that the number of arbitrary constants in the complete solution of any differential equation is the number of its order. The meaning and value of these arbitrary constants can be determined by the initial conditions when $t = 0$.

If y_0 denote the position in space of a body when $t = 0$, $C = y_0$. Again the initial velocity is U, for $\frac{dy}{dt} = Gt + U$, and when $t = 0$, the velocity is U. The acceleration $\frac{d^2y}{dt^2}$ is G, a constant. Indeed nearly all the information about motion in an unresisting medium is really contained in the innocent-looking expression $\frac{d^3y}{dt^3} = 0$, or $\dddot{y} = 0$.

The main object of differential equations, and indeed of all mathematical analysis, is to *disclose what is general in a particular case.* A differential equation is the most general way of expressing a natural law, and it is of the utmost value to practical workers to be able to write down the conditions that may obtain in any natural phenomenon as a differential equation.

For instance one observes after five successive intervals of an hour five different amounts of decomposition to have occurred in an initial amount (a) of AsH_3. One may suspect that the rate of change is proportional to the quantity present of AsH_3. Say x is the amount transformed in time t. Then this hypothesis would be expressed by the differential equation

$$\frac{dx}{dt} = k(a - x) \text{ or } \frac{dx}{a - x} = k\,dt.$$

On integrating this, we get $kt = -\lg(a - x) + C$.

Now when $t = 0$, $x = 0$; $\therefore\ C = \lg a$; $\therefore\ kt = \lg \frac{a}{a - x}$.

Now a is known, and x has been observed at stated intervals of time, so if k is found experimentally to be constant there is some reason for assuming that the hypothesis is correct under the observed conditions for AsH_3 and possibly for all monomolecular reactions.

The general form of a linear equation with constant coefficients is

$$a\frac{d^n y}{dx^n} + b\frac{d^{n-1}y}{dx^{n-1}} + \ldots + p\frac{dy}{dx} + qy = F(x),$$

where the right-hand member is called the absolute term. We shall begin by considering the simplest form of the first order in which the absolute term is absent, and see if we cannot guess some satisfactory method of solving more difficult forms.

A. **First Order,** $\boldsymbol{\frac{dy}{dx} + by = 0}$.

$$\int\frac{dy}{y} = -b\int dx; \quad \therefore \ \lg y = -bx + K,$$

or writing $\lg C$ for K, $\lg y - \lg C = -bx$,

i.e. $$\lg\frac{y}{C} = -bx \quad \text{or} \quad y = Ce^{-bx}.$$

Now if D represent the operator $\frac{d}{dx}$, the original equation may be expressed in the symbolic form by $(D+b)y = 0$, and the solution we have found to be $y = Ce^{-bx}$.

If the equation be $\frac{dy}{dx} - by = 0$, or $(D-b)y = 0$, the solution will be found to be $y = Ce^{bx}$.

For if $y = Ce^{bx}$, $\frac{dy}{dx} = bCe^{bx}$; $\therefore$ $(D-b)y = 0$.

It appears that all we have to do is to equate the expression in brackets to 0, hence find the value of D, and use this value as a factor of the exponent of Ce^{x}.

Now all algebraic symbols are operators upon units of one kind or another. The result obtained when it has operated upon the operand is the measure of the strength and nature of the operator. Here D is the operator $\frac{d}{dx}$, and Dy represents the result of an opera-

tion, as also does by; therefore $(D - b)y$ correctly represents the result of the combined operations upon y. We are assuming that the operator D can be treated as an algebraic symbol; we will presently prove that we are justified in doing so under certain conditions.

B. **Second Order, (1)** $\frac{d^2y}{dx^2} - m^2y = 0$.

If written in the symbolic form $(D^2 - m^2)y = 0$, we are tempted to try whether we may split $(D^2 - m^2)$ into its factors $(D + m)(D - m)$, and write the equation as $(D + m)(D - m)y = 0$. As two arbitrary constants are here required, since the equation is of the second order, we guess the solution to be

$$y = Me^{-mx} + Ne^{mx}.$$

Check this guess by differentiation:

$$Dy = -mMe^{-mx} + mNe^{mx},$$
$$D^2y = m^2Me^{-mx} + m^2Ne^{mx} = m^2y.$$

The guess is therefore correct, for:

(1) it satisfies the original equation, and is therefore a solution;

(2) it contains two arbitrary constants, and therefore it is the *complete* solution.

If preferred, it can be put into another form. On replacing M by $\frac{1}{2}(K - H)$ and N by $\frac{1}{2}(K + H)$, we obtain

$$\begin{aligned} y &= \tfrac{1}{2}Ke^{-mx} - \tfrac{1}{2}He^{-mx} + \tfrac{1}{2}Ke^{mx} + \tfrac{1}{2}He^{mx} \\ &= H \sinh mx + K \cosh mx. \end{aligned}$$

Can any algebraic function of D be treated in a similar way by splitting it up into factors? The procedure can only be justified by showing that the symbol D submits to the three fundamental laws of algebra. This is shown in all the books, but usually little attention is paid to this point.

(1) The *Distributive* Law, which is represented by

$$m(a - b + c + \ldots) = ma - mb + mc + \ldots$$

We know that $D(u-v+w+\ldots) = Du-Dv+Dw+\ldots$, so the symbol D is *distributive* in its operation.

(2) The *Commutative* Law, which is expressed by $ab = ba$. Now $Dcy = cDy$, but Dcy is not equal to yDc, so the symbol D is only *commutative* with regard to *constants*.

(3) The *Index* Law, represented by $a^m \times a^n = a^{m+n}$. Now to differentiate m times an expression that has already been differentiated n times, is clearly the same as differentiating the expression $m+n$ times, so $D^mD^ny = D^{m+n}y$, at any rate when m and n are positive integers, and this is all that we require at present.

Hence the operator D (when its index is a positive integer) satisfies the three fundamental laws of algebra *when the coefficients are constants.* As we are confining ourselves to differential equations with constant coefficients, we are justified in treating D in exactly the same way as an algebraic symbol.

For instance, we may expand $(D + a)^n$ by the binomial theorem or use such an expression as $f(D)$ for any algebraic function of D, such as

$$D^4 + 2D^3 - 9D^2 - 2D + 8 = f(D),$$

and we may split $f(D)$ into factors in the expression of $f(D)y$:

$$f(D)y = (D + 4)(D - 1)(D + 1)(D - 2)y + 0; \qquad \text{(p. 3)}$$

$$\therefore\ y = Ae^{-4x} + Be^{x} + Ce^{-x} + De^{2x},$$

where A, B, C, and D are the four arbitrary constants required.

It is clear then that in order to solve any linear equation with constant coefficients without an absolute term, it is only necessary to solve the equation in D; if the roots be α, β, γ, &c., the solution is

$$y = C_1e^{\alpha x} + C_2e^{\beta x} + C_3e^{\gamma x}, \text{ \&c.}$$

(2) $\dfrac{d^2y}{dt^2} + n^2y = 0$, **or** $(D^2 + n^2)y = 0$.

This is an exceedingly important equation; it is the equation of motion of a body which is under the influence

of an acceleration which is proportional to the displacement of the body from the position of rest. The motion is called *Simple Harmonic Motion* (S.H.M.); it may be represented graphically by a succession of sinuous curves (fig. 5). It expresses and explains most of the phenomena

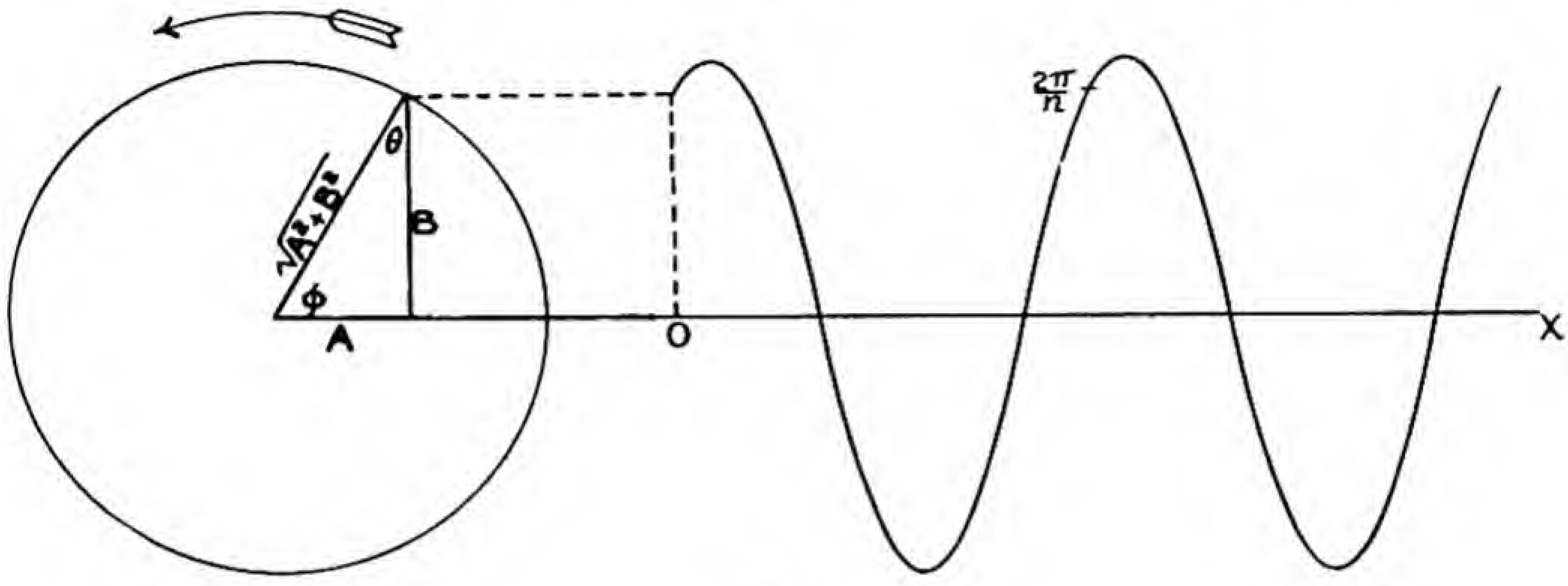

Fig. 5

of light, electric waves, sound; even the waves of the sea and the oscillations of a pendulum are rough illustrations of this kind of motion.

We use just the same method that was so successful when dealing with $(D^2 - m^2)y = 0$. Split up the expression $(D^2 + n^2)$ into factors; it is true that they will be unreal, involving $\sqrt{-1}$ or i quantities, but they will be found to be full of meaning:

$$(D^2 + n^2)y = (D + in)(D - in)y. \quad y = M\epsilon^{-int} + N\epsilon^{int}.$$

As the constants are entirely arbitrary, real or unreal, we may replace N by $\frac{1}{2}(B - iA)$ and M by $\frac{1}{2}(B + iA)$.

Then $$y = \frac{B}{2}(\epsilon^{int} + \epsilon^{-int}) - \frac{iA}{2}(\epsilon^{int} - \epsilon^{-int}).$$

Multiply $-\frac{iA}{2}$ by $\frac{i}{i}$, and we obtain $+\frac{A}{2i}$, so

$$y = \frac{A}{2i}(\epsilon^{int} - \epsilon^{-int}) + \frac{B}{2}(\epsilon^{int} + \epsilon^{-int}), \text{ or } A \sin nt + B \cos nt,$$

where the nt of course is measured in radians.

We may put this into another and usually a more convenient form by putting

$$C=\sqrt{A^2+B^2},\ \cos\phi=\frac{A}{\sqrt{A^2+B^2}},\ \text{and}\ \sin\phi=\frac{B}{\sqrt{A^2+B^2}},$$

where ϕ is an arbitrary angle whose tangent is $\frac{B}{A}$.

Then $y = C(\sin nt \cos\phi + \cos nt \sin\phi)$ or $C \sin(nt + \phi)$.

Here C represents the amplitude of the wave, while ϕ represents the epoch or phase in which the wave motion is, when $t = 0$, and $\frac{2\pi}{n}$ is the period (fig. 5).

An alternative form for $C\sin(nt+\phi)$ or $C\sin\left(nt+\tan^{-1}\frac{B}{A}\right)$ is $C\cos(nt-\theta)$ or $C\cos\left(nt-\cot^{-1}\frac{B}{A}\right)$.

For $$\sin(nt+\phi)=\sin\left\{nt+\frac{\pi}{2}-\left(\frac{\pi}{2}-\phi\right)\right\}$$
$$=\cos(nt-\theta)\ \text{or}\ \cos\left(nt-\cot^{-1}\frac{B}{A}\right),$$

where $$\theta=\frac{\pi}{2}-\phi.$$

In the diagram the curve traced is $C \sin(nt + \varphi)$; it can never rise above the axis OX higher than C, or sink lower than $-C$; the interval between two adjacent crests, or the wave-length (λ) must correspond to one complete revolution (i.e. 2π) of the radius, indicated by $\sqrt{A^2 + B^2}$ or C. If the abscissa represent the time, and T is the *period* of the wave, or the time taken in making one complete oscillation, at successive intervals of distance T, the wave disturbance will be in exactly the same phase; so the ordinate of the wave will be exactly the same height. When $nt = 0$, 2π or $2m\pi$, the ordinate will be $C \sin\varphi$, in this case $C \sin\frac{\pi}{3}$, as indicated by the dotted ordinate.

Now if v be the velocity of transmission of the wave, $\lambda = vT$, so the equation can be given as $C \sin\left(\frac{2\pi}{T}t + \varphi\right)$ or $C \sin\left(\frac{2\pi}{\lambda}vt + \varphi\right)$, i.e. $C \sin\left(\frac{2\pi}{\lambda}x + \varphi\right)$. It is clear then that the wave is in the same

phase of motion $\left(\frac{\pi}{3}\right)$ whenever $\frac{t}{T}$ or $\frac{x}{\lambda}$ have integral values, and that the ordinate of the curve is then $C \sin \varphi$. Hence the curve is said to be periodic in both space and time.

Note that the expression nt denotes a pure number, for an angle is measured by the ratio of the arc to the radius, so it is of no dimensions; but t is of one dimension in time, so the constant n must be of -1 dimension in time, and as we have found above n is $\frac{2\pi}{T}$, or an angular velocity.

A graph of S.H.M. is easily made. Describe a circle of radius C; draw a horizontal line through the centre and produce it to the right. Now, taking the upper and lower quadrants, divide each into three equal parts, so that each part of the vertical semicircle subtends an angle of $\frac{\pi}{6}$ at the centre of the circle. Through each of these points draw faint pencil lines parallel to the axis of x. The curve will meet each of these lines twice in one complete wave-length, except the line that indicates the crests of the waves and that which indicates the troughs of the waves. Divide that part of the axis of x that represents a wave-length or a period into twelve equal parts and erect ordinates to meet the appropriate pencil line, and so twelve points will be given through which the curve may be traced. Distances from O along OX represent t.

EXAMPLES.

(1) $$\frac{d^4y}{dx^4} - p^4y = 0, \text{ or } (D^2 + p^2)(D^2 - p^2)y = 0.$$

$$\therefore\ y = A \sin px + B \cos px + C \sinh px + D \cosh px.$$

(2) $$\{D^4 + (p^2 - q^2)D^2 - p^2q^2\}y = 0.$$

$$\therefore y = A \sin px + B \cos px + C \sinh qx + D \cosh qx.$$

Both these equations turn up in engineering practice and in the theories of Sound and Heat.

(3) $$\frac{d^2y}{dt^2} + 2p\frac{dy}{dt} + n^2y = 0, \text{ or } (D^2 + 2pD + n^2)y = 0.$$

(i) $p > n$.

The roots are real, say α and β, and the solution is

$$y = Me^{\alpha t} + Ne^{\beta t}.$$

(ii) $p = n$.

The roots are identical, each being equal to $(-p)$, but we must provide two arbitrary constants. In such a case the following device is employed, $y = C_0 e^{-pt} + C_1 t e^{-pt}$.

A justification of this procedure will be given subsequently, but for the moment the reader must content himself with finding that he can regain the original equation by using the values given for y.

Similarly, if we are dealing with an equation in D with n identical roots such as $(D - a)^n y = 0$, the solution is

$$y = C_0 e^{at} + C_1 t e^{at} + \ldots C_{n-1} t^{n-1} e^{at};$$

if with m pairs of identical unreal roots such as $(D^2 + n^2)^m$,

$$y = C_0 \sin(nt + \phi_0) + C_1 t \sin(nt + \phi_1) + \ldots$$

(iii) $p < n$.

The roots are unreal, being $-p \pm \sqrt{p^2 - n^2}$ or $-p \pm im$ where $m = \sqrt{n^2 - p^2}$. The solution will therefore be

$$y = Me^{-pt+imt} + Ne^{-pt-imt} \quad \text{or} \quad e^{-pt}(M\varepsilon^{imt} + N\varepsilon^{-imt});$$

i.e. $\quad e^{-pt}(A \sin mt + B \cos mt)$ or $e^{-pt} C \sin(mt + \phi)$.

This equation is usually called that of *Damped Oscillations*. The diagram (fig. 6) shows a strong damping effect on the S.H.M., which is indicated by spaced lines.

It will be noticed that in S.H.M. or a " free oscillation ", if the period be denoted by T_0, we have $n = \dfrac{2\pi}{T_0}$, but in a damped oscillation n becomes m or $\sqrt{n^2 - p^2}$, so T_0 becomes T_1 where $T_1 = \dfrac{2\pi}{m}$ or $\dfrac{2\pi}{\sqrt{n^2 - p^2}}$, consequently *damping increases the period* of oscillation from $\dfrac{2\pi}{n}$ to $\dfrac{2\pi}{m}$.

Also the amplitude of each vibration is in a constant ratio to that of the next. In practice when observing the oscillations of a pendulum or a galvanometer needle the length of one swing is noted: i.e. the distance or angle from an extreme position on one side of zero to the next

extreme position on the other side of zero, so observations are taken at each half-period of a complete oscillation, or if more convenient at the first swing and the eleventh. The ratio in which the amplitude diminishes in one swing is called the *damping ratio*, and the Napierian logarithm of this ratio is called the *logarithmic decrement*, and is $-\frac{1}{2}pT$.

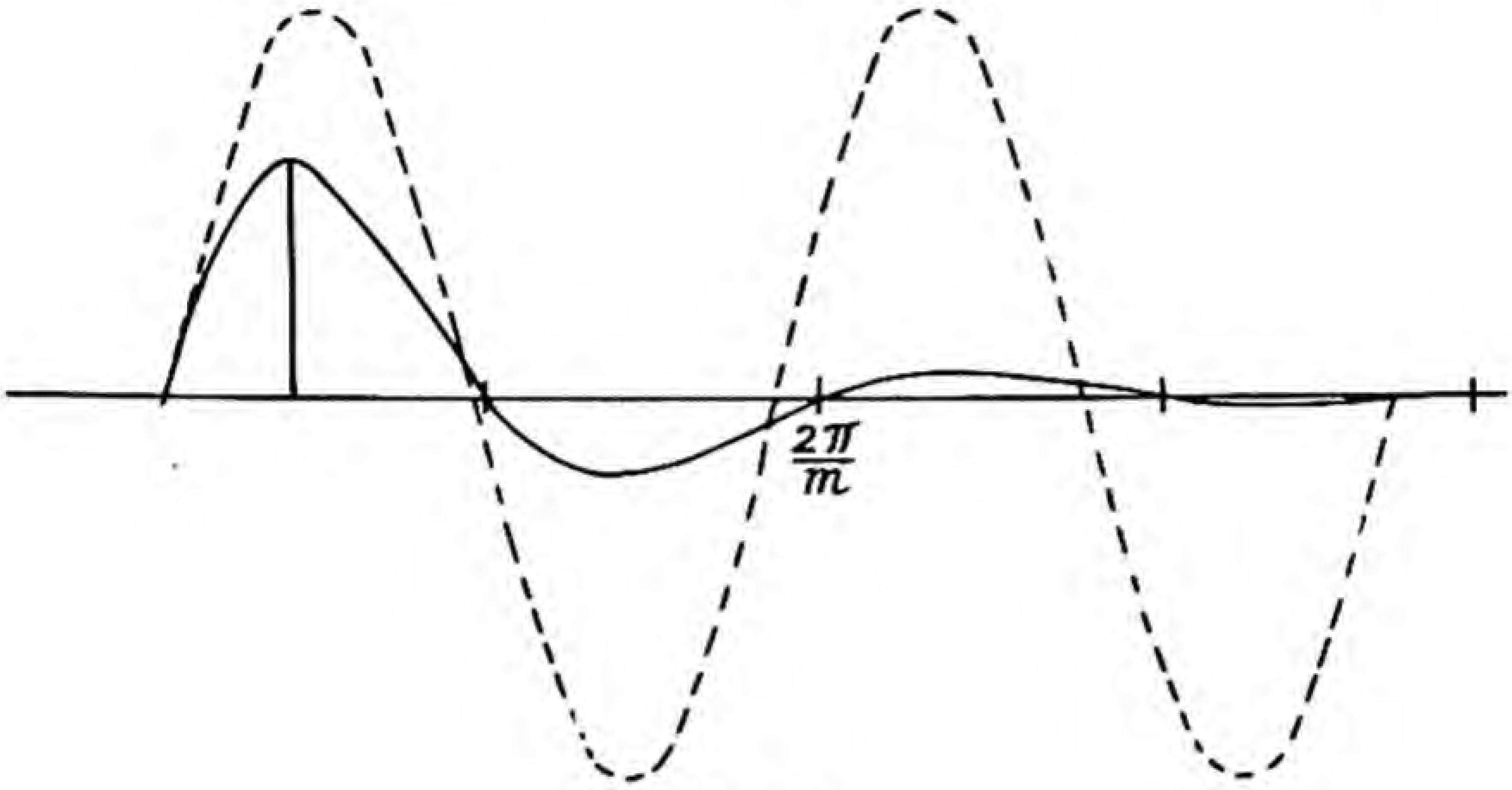

Fig. 6

It should be noted that whenever $p < n$ in the differential equation $(D^2 + 2pD + n^2)y = 0$, a *periodic* motion occurs, but when $p > n$ an *aperiodic* motion results.

A warning must now be given about using this equation in unsuitable cases; we have assumed that the damping effect is due to some cause ($2p$) that varies as the velocity, and this is often an unwarrantable assumption. The resistance of the air to projectiles, for instance, at high speeds varies as v^2, v^3 or even higher powers of the velocity. Engineers, however, tell us that the friction of well-lubricated surfaces appears to vary almost in proportion to the velocity, and fortunately in electrical work laboratory results agree closely in most cases with the formulæ given above, but great caution must be exercised before applying them to new conditions.

Equations with an Absolute Term.

For simplicity take an equation of the first order in which the absolute term is a constant, say c;

$$\frac{dy}{dx} + by = c, \text{ or } (D + b)y = c.$$

Clearly, if we put $y = z + \frac{c}{b}$, we get $\frac{dz}{dx} + bz + c = c$, or $(D + b)z = 0$,

$$\text{so } z = Ce^{-bx}, \quad y = Ce^{-bx} + \frac{c}{b}.$$

In all cases the solution of the left side equated to 0 is called the *Complementary Function* (C.F.), whereas the term $\frac{c}{b}$ is the simplest possible solution of the whole equation, and is called the *Particular Integral* (P.I.). Obviously $(D + b)\frac{c}{b} = 0 + c$.

It will be seen that for every equation with an absolute term one has only to find the complementary function, which may be denoted by u, and then find or guess the simplest possible P.I. which will satisfy the original equation; the complete solution is then $y = u + w$, where w is the P.I., for all the necessary arbitrary constants are given in the expression for u.

The symbol D^{-n} will denote n successive integrations with the omission of the usual arbitrary constants, and we may with propriety transform $\{f(D)\}^{-1}$ into any suitable equivalent form recognized by algebra. For instance $(D^2 + \overline{a + b}\,D + ab)^{-1}$ may be transformed into partial fractions $\frac{1}{a - b}\left(\frac{1}{D + b} - \frac{1}{D +}\right)$.

I. The absolute term is an algebraic expression.

The procedure is best shown by an example; e.g.

$$(D^2 + n^2)y = 3t^2 - 8; \text{ here } D \text{ represents } \frac{d}{dt}.$$

We have already found the C.F. or u to be $C \sin(nt + \phi)$, and we have only to find the P.I. or w.

$$(D^2 + n^2)w = 3t^2 - 8,$$

or $$w = (n^2 + D^2)^{-1}(3t^2 - 8),$$

or $$w = (n^{-2} - n^{-4}D^2 + n^{-6}D^4 - \ldots)(3t^2 - 8),$$

or $$w = \frac{3t^2}{n^2} - \frac{8}{n^2} - \frac{6}{n^4}.$$

It will be noted that the expansion of $\{f(D)\}^{-1}$ in ascending powers of D need not be carried farther than the highest power of the independent variable in the absolute term, as all subsequent terms vanish.

The complete solution is $y = u + w$,

i.e. $$y = C\sin(nt + \phi) + \frac{3t^2}{n^2} - \frac{8}{n^2} - \frac{6}{n^4}.$$

Of course the same method might have been used in the first example, where the absolute term was a constant.

On referring to the table of successive differentiation (p. 56), it will be seen that $D^n e^{ax} = a^n e^{ax}$, and it is clear from the definition of D^{-n} that this is equally true when n is negative. If $f(D)$ can be represented by a series $\Sigma A_n D^n$, we have

$$f(D)e^{ax} = (\Sigma A_n D^n)e^{ax} = (\Sigma A_n a^n)e^{ax} = f(a)e^{ax}.$$

Hence the effect of the operator $f(D)$ or $\dfrac{1}{f(D)}$ on e^{ax} can be obtained on merely replacing D by a, e.g.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 8e^{4x} \text{ or } (D-2)(D-3)y = 8e^{4x}.$$

Then the P.I. or

$$w = \frac{1}{(D-2)(D-3)}8e^{4x} = \frac{1}{(4-2)(4-3)}8e^{4x} = 4e^{4x}.$$

The C.F. or u is $Ae^{2x} + Be^{3x}$;

$$\therefore\ y = u + w = Ae^{2x} + Be^{3x} + 4e^{4x}.$$

Similarly, the operation

$$f(D)a^{mx} = f(m\lg a)e^{mx\lg a} = f(m\lg a)a^{mx} \text{ for } a = e^{\lg a}.$$

II. The operation $f(D)e^{at}F(t) = e^{at}f(D+a)F(t)$.

Let T be any function of t such as $F(t)$. Then, since $D^n e^{at} = a^n e^{at}$, we have by Leibnitz's theorem

$$D^n(e^{at}T) = e^{at}(a^n T + {}_nC_1 a^{n-1}DT + {}_nC_2 a^{n-2}D^2T + \ldots D^nT),$$

or $D^n(e^{at}T) = e^{at}(D+a)^n T$. The result may also be used when n is a negative integer.

The expression e^{at} can always be transferred from the right to the left side of $f(D)$, if D is replaced by $D+a$. This is a most valuable theorem; it will give the solution of all the difficulties we have so far encountered.

On p. 90, when dealing with the C.F., a difficulty occurred in the solution of $(D^2 + 2pD + p^2)y = 0$.

The complete solution of $(D+p)y = 0$ is $y = Ce^{-pt}$, so what is required is the solution of $(D+p)y = Ce^{-pt}$.

The P.I. or w will be $\dfrac{1}{D+p}Ce^{-pt}$ or $Ce^{-pt}\dfrac{1}{D}(1) = Ce^{-pt}t$.
We may expect then the complete solution of the original equation to be given by regarding this C as a different constant, so the complete solution is

$$y = C_1e^{-pt} + C_2te^{-pt}.$$

Suppose that in the previous section the equation given were $(D-2)(D-3)y = 8e^{3x}$, there would be a difficulty about the P.I.

$$w = \frac{1}{(D-2)(D-3)}8e^{3x} = \frac{1}{(3-2)(D-3)}8e^{3x} = \frac{1}{D-3}8e^{3x}$$

$$= 8e^{3x}\frac{1}{D}1 = 8e^{3x}x,$$

so the complete solution would be

$$y = Ae^{2x} + Be^{3x} + 8xe^{3x}.$$

III. The operation $f(D^2)\begin{matrix}\sin\\ \cos\end{matrix}(nt) = f(-n^2)\begin{matrix}\sin\\ \cos\end{matrix}(nt)$.

(i) E.g. $(D^3 + D^2 + D + 1)y = \sin 2t$.

The C.F. or u is found from $(D^2 + 1)(D + 1)u = 0$, or $u = C_1 \sin(t + \phi) + C_2 e^{-t}$.

The P.I. or $w = (D^3 + D^2 + D + 1)^{-1} \sin 2t$,

$$w = \frac{D-1}{D^4 - 1} \sin 2t = \frac{D-1}{(-4)^2 - 1} \sin 2t = \tfrac{1}{15}\,(2 \cos 2t - \sin 2t),$$

$$y = C_1 \sin(t + \phi) + C_2 e^{-t} + \tfrac{1}{15}\,(2 \cos 2t - \sin 2t).$$

(ii) $(D^2 + n^2)y = k \sin pt$,

$$u = C \sin(nt + \phi), \quad w = \frac{k}{n^2 - p^2} \sin pt,$$

$$y = C \sin(nt + \phi) + \frac{k}{n^2 - p^2} \sin pt.$$

This introduces the subject of Forced Oscillations, which is of enormous importance in physics. If the ideally simplest case be considered in which $p = n$, and all resistance be neglected, it is clear that the above expression would make the amplitude infinite, for $\dfrac{k}{n^2 - n^2} = \infty$, which is absurd.

The most satisfactory way of dealing with this difficulty when the absolute term is of the same period as the C.F. or $t \sin nt$, $t^2 \cos nt$, &c., is the following:

(iii) $$(D^2 + n^2)y = k \sin nt + k \cos nt.$$

As $\epsilon^{int} = \cos nt + i \sin nt$, we may replace

$$\frac{k}{D^2 + n^2} (\sin nt + \cos nt) \text{ by } \frac{k}{D^2 + n^2} \epsilon^{int},$$

provided that we only make use of the unreal terms for $\sin nt$, denoting them by iw_2 and the real terms w_1, for $\cos nt$.

$$w_1 + iw_2 = \epsilon^{int} \frac{k}{(D+in)^2+n^2} 1 = \epsilon^{int} \frac{k}{2in+D} \frac{1}{D}(1),$$

$$= k\epsilon^{int} \frac{1}{2in+D} t = k\epsilon^{int}(2in+D)^{-1}t,$$

$$= k\epsilon^{int}\left(\frac{1}{2in} - \frac{D}{(2in)^2} \cdots\right)t$$

$$= k(\cos nt + i \sin nt)\left(\frac{t}{2in} + \frac{1}{4n^2}\right);$$

$$\therefore \; w_1 = \frac{k \cos nt}{4n^2} + \frac{kt \sin nt}{2n}$$

and $iw_2 = \dfrac{kt \cos nt}{2in} + \dfrac{ki \sin nt}{4n^2}$, so $w_2 = \dfrac{k \sin nt}{4n^2} - \dfrac{kt \cos nt}{2n}$.

Now as $(D^2 + n^2)u = 0$, $u = A \sin nt + B \cos nt$, so the first part of $w_1 \left(\text{i.e. } \dfrac{k \cos nt}{4n^2}\right)$ may be included in the arbitrary constant B; similarly, the first part of $w_2 \left(\text{i.e. } \dfrac{k \sin nt}{4n^2}\right)$, if included in the arbitrary constant A, may be neglected.

So $(D^2 + n^2)y = k \sin nt + k \cos nt$ gives

$$y = A \sin nt + B \cos nt - \frac{kt \cos nt}{2n} + \frac{kt \sin nt}{2n}.$$

If the equation given be $(D^2 + n^2)y = k \sin nt$,

$$y = \sqrt{A^2 + B^2} \sin\left(nt + \tan^{-1}\frac{B}{A}\right) - \frac{kt \cos nt}{2n},$$

but the last term may be written $\dfrac{kt}{2n} \sin\left(nt - \dfrac{\pi}{2}\right)$.

Consequently this term (which may be regarded as due to the external force) increases with the time, but introduces

a lag of $\frac{\pi}{2}$; in a short time the oscillation is increased enormously.

If the absolute term be $k \cos nt$, the solution may be written

$$y = \sqrt{A^2 + B^2} \cos\left(nt - \cot^{-1}\frac{B}{A}\right) + \frac{kt}{2n} \sin nt,$$

or $$\sqrt{A^2 + B^2} \cos\left(nt - \cot^{-1}\frac{B}{A}\right) + \frac{kt}{2n} \cos\left(nt - \frac{\pi}{2}\right).$$

The same method should be used if any powers of the independent variable occur as factors of the sine or cosine.

(iv) $$(D^2 + n^2)y = t \cos t + t \sin t.$$

The P.I. is $\frac{1}{(D^2 + n^2)} (\cos t + \sin t)t$.

Now $\frac{1}{D^2 + n^2} \epsilon^{it}t$, or

$$\epsilon^{it} \frac{1}{(D + i)^2 + n^2} t = \epsilon^{it} \frac{1}{(n^2 - 1) + (2iD + D^2)} t$$

$$= (\cos t + i \sin t)\left(\frac{t}{n^2 - 1} - \frac{2i}{(n^2 - 1)^2}\right)$$

$$= \frac{t \cos t}{n^2 - 1} - \frac{2i \cos t}{(n^2 - 1)^2} + \frac{it \sin t}{n^2 - 1} + \frac{2 \sin t}{(n^2 - 1)^2}.$$

But $i \sin t$, being unreal, must be represented by

$$iw_2 \quad \text{or} \quad \frac{it \sin t}{n^2 - 1} - \frac{2i \cos t}{(n^2 - 1)^2}.$$

So for $(D^2 + n^2)y = t \sin t$ we obtain

$$y = C \sin(nt + \phi) + \frac{t \sin t}{n^2 - 1} - \frac{2 \cos t}{(n^2 - 1)^2},$$

and $(D^2 + n^2)y = t \cos t + t \sin t$ gives

$$y = C \sin(nt + \phi) + \frac{t(\sin t + \cos t)}{n^2 - 1} + \frac{2(\sin t - \cos t)}{(n^2 - 1)^2}.$$

(v) $$(D^2 + n^2)y = t \sin nt.$$

This can be solved in a precisely similar way.

$$\epsilon^{int} \frac{1}{D + 2in} \frac{1}{D} t = \epsilon^{int} \frac{1}{D + 2in} \frac{t^2}{2}$$

$$= (\cos nt + i \sin nt) \left(\frac{1}{2in} - \frac{D}{(2in)^2} + \frac{D^2}{(2in)^3} \right) \frac{t^2}{2}.$$

$$\therefore\ iw_2 = \frac{t^2 \cos nt}{4in} + \frac{it \sin nt}{4n^2} + \frac{\cos nt}{8(in)^3};$$

so $$w_2 = -\frac{t^2 \cos nt}{4n} + \frac{t \sin nt}{4n^2} + \frac{\cos nt}{8n^3}.$$

If the external force ($k \sin nt$ or $k \cos nt$) be of the same form as that of the free oscillation, the result is a great oscillation of the period $\frac{2\pi}{n}$. It is out of step with that of the free oscillation unless by chance $+ \tan^{-1} \frac{B}{A} = -\frac{\pi}{2}$, or $-\cot^{-1} \frac{B}{A} = -\frac{\pi}{2}$, but that is of slight importance as the external force increasing with the time exerts a preponderating influence on the disturbance.

The phenomena of resonance in acoustics, of wireless reception, of oscillation in wireless, of heavy rolling of one ship at sea when others are relatively free from rolling, are all examples of this type. If the period of the wave at sea corresponds with the natural period of a boat or is a simple multiple of that of a certain ship, that boat or ship will roll heavily while other vessels of a different period will remain practically undisturbed. Iron bridges have broken down when soldiers have been marching over them in step with the natural period of vibration of the bridge. The "beats" in music and "boating" in wireless are produced by two trains of waves that differ slightly in frequency. For simplicity we will suppose the amplitude (C) of each to be the same.

Then $$C \sin nt + C \sin(nt + ht) = 2C \cos \frac{ht}{2} \sin\left(nt + \frac{ht}{2}\right).$$

In music this will represent a single tone between the frequencies of the two trains, and the amplitude varies from $2C$ through 0 to $-2C$, and so on. Suppose that n denote a frequency of 256 a second, and $n + h$ a frequency of 260 a second, then in a quarter of a second

one note will have made 65 complete vibrations and the other 64 vibrations; so there will be 4 (h) beats a second, and the intensity of the sound will go through its whole range of values in a quarter of a second. If the amplitude of both vibrations be not the same, the pitch will vary; when the sound is louder, the pitch is between that of either, but when it is faint, the pitch is rather flatter than either of the original notes.

$$\text{IV. } (aD+b)\begin{matrix}\sin\\ \cos\end{matrix}(qt+\theta)=\sqrt{a^2q^2+b^2}\begin{matrix}\sin\\ \cos\end{matrix}\left(qt+\theta+\tan^{-1}\frac{aq}{b}\right).$$

$$(aD-b)\begin{matrix}\sin\\ \cos\end{matrix}(qt+\theta)=-\sqrt{a^2q^2+b^2}\begin{matrix}\sin\\ \cos\end{matrix}\left(qt+\theta-\tan^{-1}\frac{aq}{b}\right).$$

$$\frac{1}{aD+b}\begin{matrix}\sin\\ \cos\end{matrix}qt=\frac{1}{\sqrt{a^2q^2+b^2}}\begin{matrix}\sin\\ \cos\end{matrix}\left(qt-\tan^{-1}\frac{aq}{b}\right).$$

$$\frac{1}{aD-b}\begin{matrix}\sin\\ \cos\end{matrix}qt=\frac{-1}{\sqrt{a^2q^2+b^2}}\begin{matrix}\sin\\ \cos\end{matrix}\left(qt+\tan^{-1}\frac{aq}{b}\right).$$

These expressions frequently turn up in electrical and engineering work; two of them are given by Perry. It is seen that $(aD+b)$ induces a *lead* of $\tan^{-1}\frac{aq}{b}$, and multiplies the amplitude by $\sqrt{a^2q^2+b^2}$; the inverse operation induces a *lag*, and divides the amplitude by the square root factor. The equations are quite easily obtained, but are handy for reference.

Equations that can be reduced to a Linear Form with Constant Coefficients.

(I) $\frac{dy}{dx}+Py=Qy^n$, where P and Q are constants.

Divide by y^n, then $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$;

et $y^{1-n}=z$, then $(1-n)y^{-n}\frac{dy}{dx}=\frac{dz}{dx}$;

$$\therefore\ \frac{dz}{dx}+(1-n)Pz=(1-n)Q,$$

$$z \text{ or } y^{1-n}=Ce^{\overline{n-1}Px}+\frac{Q}{P}.$$

(II) Sometimes a change of the independent variable will simplify a differential equation, or even reduce it to the required linear form.

(1) First suppose that x is a known function of t or θ.

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}, \quad \therefore \frac{d}{dx} = \frac{1}{\dfrac{dx}{dt}} \frac{d}{dt}.$$

So
$$\frac{d^2y}{dx^2} = \frac{1}{\dfrac{dx}{dt}} \frac{d}{dt}\left[\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}\right] = \frac{\dfrac{d^2y}{dt^2}\dfrac{dx}{dt} - \dfrac{d^2x}{dt^2}\dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3};$$

e.g.
$$(a^2 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 = f(x),$$

where $x = a\cos\theta$.

Here $\dfrac{dx}{d\theta} = -a\sin\theta$, and $\dfrac{d^2x}{d\theta^2} = -a\cos\theta$.

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{-a\sin\theta}; \quad \frac{d^2y}{dx^2} = \frac{-a\sin\theta\dfrac{d^2y}{d\theta^2} + a\cos\theta\dfrac{dy}{d\theta}}{(-a\sin\theta)^3}.$$

So $f(x)$ becomes $a^2\sin^2\theta\dfrac{d^2y}{dx^2} - a\cos\theta\dfrac{dy}{dx} + y = 0$,

or
$$\frac{d^2y}{d\theta^2} - \cot\theta\frac{dy}{d\theta} + \cot\theta\frac{dy}{d\theta} + y = 0.$$

$$\therefore y = A\sin\theta + B\cos\theta \quad \text{or} \quad C\sin(\theta + \phi).$$

(2) If y is made the independent variable instead of x, put $t = y$; then we have

$$\frac{dy}{dt} = 1, \text{ and } \frac{d^2y}{dt^2} = 0.$$

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}},\quad \frac{d^2y}{dx^2}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}.$$

E.g. $$\frac{d^2y}{dx^2}-2p\left(\frac{dy}{dx}\right)^2-n^2x\left(\frac{dy}{dx}\right)^3=0.$$

This becomes the familiar equation

$$-\frac{1}{\left(\frac{dx}{dy}\right)^3}\left(\frac{d^2x}{dy^2}+2p\frac{dx}{dy}+n^2x\right)=0,$$

where x is aperiodic if $p>n$, but a damped oscillation if $p<n$,

$$x=Ce^{-py}\sin(my+\phi).$$

These two illustrations will indicate the sort of cases in which such a procedure may be of use.

(III) The form

$$A_0x^n\frac{d^ny}{dx^n}+A_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots+A_ny=f(x)$$

can always be readily transformed into a linear equation with constant coefficients, if A_0, A_1, &c., be constants.

Let $x=e^t$; then $\frac{dx}{dt}=e^t=x$.

$$\frac{dy}{dt}=\frac{dx}{dt}\frac{dy}{dx}=e^t\frac{dy}{dx}=x\frac{dy}{dx}.$$

Also, differentiating first, then multiplying by x,

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)=x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2},$$

or $$x^2\frac{d^2y}{dx^2}=\left(x\frac{d}{dx}-1\right)\left(x\frac{dy}{dx}\right).$$

Similarly,

$$x\frac{d}{dx}\left(x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}\right)=x\left(\overline{n-1}\,x^{n-2}\frac{d^{n-1}y}{dx^{n-1}}+x^{n-1}\frac{d^ny}{dx^n}\right).$$

$$\therefore\ x^n\frac{d^ny}{dx^n}=\left(x\frac{d}{dx}-n+1\right)\left(x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}\right).$$

But it has been shown above that the operators $x\dfrac{d}{dx}$ and $\dfrac{d}{dt}$ are equivalent, so that D may be used for either. It is more usual to employ the symbol ϑ in this case, but we shall continue the use of D.

Hence

$$x^n\frac{d^ny}{dx^n}=(D-n+1)\,(D-n+2)\ldots(D-1)Dy,$$

or $$=D(D-1)\,(D-2)\ldots(D-n+1)y.$$

(1) E.g. $$\frac{d^3y}{dx^3}+\frac{3}{x}\,\frac{d^2y}{dx^2}-\frac{1}{x^2}\,\frac{dy}{dx}+\frac{y}{x^3}=2,$$

or $$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=2x^3.$$

Let $x=e^t$; then

$$D(D-1)\,(D-2)y+3D(D-1)y-Dy+y=2e^{3t},$$

or $$(D-1)\,(D^2-2D+3D-1)y=2e^{3t},$$

or $$(D-1)\,(D^2+D-1)y=2e^{3t};$$

$$u=Ae^t+e^{-\frac{1}{2}t}\left(Be^{\frac{\sqrt{5}}{2}t}+Ce^{-\frac{\sqrt{5}}{2}t}\right).$$

And $(D-1)\,(D^2+D-1)w=2e^{3t}$.

$$w=\frac{1}{D^3-2D+1}\,2e^{3t}=\frac{2e^{3t}}{27-6+1}=\frac{1}{11}\,e^{3t};$$

$$y=Ae^t+e^{-\frac{1}{2}t}\left(Be^{\frac{\sqrt{5}}{2}t}+Ce^{-\frac{\sqrt{5}}{2}t}\right)+\frac{1}{11}\,e^{3t},$$

or $$y = Ax + \frac{1}{\sqrt{x}}\left(Bx^{\frac{\sqrt{5}}{2}} + Cx^{-\frac{\sqrt{5}}{2}}\right) + \frac{1}{11}x^3.$$

(2) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + n^2y = \lg x.$

Let $x = e^t$; $\therefore\ t = \lg x.$

$D(D-1)y + (D+n^2)y = t$ or $(D^2+n^2)y = t;$

$$u = A\sin nt + B\cos nt, \text{ or } C\sin(nt+\phi),$$

$$w = \frac{1}{D^2+n^2}t = \frac{1}{n^2}\left(1 - \frac{D^2}{n^2} + \ldots\right)t - = \frac{t}{n^2},$$

$$y = C\sin(n\lg x + \phi) + \frac{\lg x}{n^2}.$$

Textbooks must be consulted for methods for differential equations with coefficients that involve the variables, except in the special cases mentioned above. However, attention to the following well-known results will often give the solution in simple cases:

$$\int u\,dv + \int v\,du = uv + C.$$

$$\int \frac{v\,du - u\,dv}{v^2} = \frac{u}{v} + C.$$

E.g. $$(x - y^2)dx + 2xy\,dy = 0.$$

Let $y^2 = v$; then $2y\,dy = dv$.

$x\,dx + x\,dv - v\,dx = 0$. Divide by x^2 and integrate.

$$\int\frac{dx}{x} + \int\frac{x\,dv - v\,dx}{x^2} = C, \text{ i.e. } \lg x + \frac{v}{x} = \lg x + \frac{y^2}{x} = C.$$

Some Partial Differential Equations with Constant Coefficients.

We shall limit ourselves to the consideration of those partial equations in which only derivatives of the nth order occur, as it is only in such cases that the following simple procedure will obtain the required solution. We have seen that the complete solution of an ordinary differential equation of the nth order must contain n arbitrary constants; but the complete solution of a partial differential equation of the nth order must contain n arbitrary functions. This will be evident from the following examples.

(1) Let $z = f(y + 3x)$.

$$\frac{\partial z}{\partial x} = 3f'(y + 3x); \quad \frac{\partial z}{\partial y} = f'(y + 3x);$$

$$\therefore \quad \frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 0.$$

So from an entirely arbitrary function of x and y a partial differential equation of the first order has been derived.

If $\frac{\partial z}{\partial x}$ be denoted by D, and $\frac{\partial z}{\partial y}$ by D_1, this may be expressed symbolically by

$$(D - 3D_1)z = 0.$$

(2) Let $z = f(y + 3x) + \varphi(y - 2x)$.

Then $$\frac{\partial z}{\partial x} = 3f'(y + 3x) - 2\varphi'(y - 2x),$$

or $$Dz = 3f' - 2\varphi';$$

and $$\frac{\partial z}{\partial y} = f'(y + 3x) + \varphi'(y - 2x),$$

or $$D_1 z = f' + \varphi'.$$

Hence $\frac{\partial^2 z}{\partial x^2}$ or $D^2 z = 9f'' + 4\varphi''$,

$\frac{\partial^2 z}{\partial x \partial y}$ or $DD_1 z = 3f'' - 2\varphi''$, and $\frac{\partial^2 z}{\partial y^2}$ or $D_1{}^2 z = f'' + \varphi''$;

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0,$$

or $$(D^2 - DD_1 - 6D_1{}^2)z = 0.$$

Here from two entirely arbitrary functions of x and y a partial differential equation of the second order has been derived in which neither of the functions appears.

Now when the coefficients of the partial derivatives are *constants*, all these derivatives obey the three fundamental laws of algebra: the Distributive, the Commutative, and the Index laws; so that these operators may be treated exactly like algebraic expressions.

Now equation (1) or $(D - 3D_1)z = 0$ resembles the ordinary differential equation of the first order $(D - a)y = 0$, the solution of which is $y = C\varepsilon^{ax}$, where both a and C are independent of x. It has just been shown that instead of C we must write an arbitrary function which must be independent of x, so for C write $f(y)$. Then by analogy we are tempted to write the solution as $z = \varepsilon^{3xD_1} f(y)$. But this is merely the symbolic form of Taylor's theorem for

$$f(y + 3x) = \left\{1 + 3xD_1 + \frac{(3x)^2}{\underline{|2}} D_1^2 + \ldots\right\} f(y),$$

so whenever Taylor's theorem does not fail, the solution of $(D - nD_1)z = 0$ will be $z = f(y + nx)$.

The *Practical Rule* for this type of equation, say $(2D + 3D_1)z = 0$, is this: in the bracket replace D by y and D_1 by $-x$; write z as equal to the function so formed; e.g. in the above $z = f(2y - 3x)$.

In equation (2),

$$(D^2 - DD_1 - 6D_1^2)z = 0, \text{ or } (D - 3D_1)(D + 2D_1)z = 0.$$

$$z = f(y + 3x) + \varphi(y - 2x).$$

If there be two or more identical roots, as for instance

$$(2D - D_1)^3 z = 0,$$

the procedure is analogous to that employed in ordinary equations:

$$z = f_1(2y + x) + xf_2(2y + x) + x^2f_3(2y + x).$$

The equation $(D^2 + 2pDD_1 + n^2D_1^2)z = 0$ is easily solved if $p > n$,

$$z = f_1(y - px - \sqrt{p^2 - n^2}\,x) + f_2(y - px + \sqrt{p^2 - n^2}\,x),$$

but if $p < n$, each function involves unreal values.

When $p < n$, let $m = \sqrt{n^2 - p^2}$; then

$$z = f_1(y - px - imx) + f_2(y - px + imx).$$

When m is integral, the following ingenious procedure given by Mr. Johnson will express the solution in real functions of x and y. It is based on the analogy of two other functions, φ_1 and φ_2, where $\varphi_1(t) = t^m$, and $\varphi_2(t) = \varepsilon^t$.

An example will explain the method:

$(D^2 + 8DD_1 + 25D_1^2)z$ or $(D + 4D_1 + i3D_1)(D + 4D_1 - i3D_1)z = 0$;

then $$z = f_1(y - 4x - i3x) + f_2(y - 4x + i3x),$$

or writing P for $y - 4x$, $z = f_1(P - i3x) + f_2(P + i3x)$.

Assume that $\varphi_1 = f_1 + f_2$, and that $i\varphi_2 = f_1 - f_2$,

so $$f_1 = \tfrac{1}{2}(\varphi_1 + i\varphi_2) \text{ and } f_2 = \tfrac{1}{2}(\varphi_1 - i\varphi_2).$$

In this case $m = 3$, so $\varphi_1(t) = t^3$;

$$\tfrac{1}{2}[(P - i3x)^3 + (P + i3x)^3] + \frac{i}{2}(\varepsilon^{P-i3x} - \varepsilon^{P+i3x})$$
$$= \tfrac{1}{2}(P^3 - i9xP^2 + i^2 27x^2P - 27i^3x^3)$$
$$+ \tfrac{1}{2}(P^3 + i9xP^2 + i^2 27x^2P + 27i^3x^3) + \varepsilon^P \frac{i}{2}(\varepsilon^{-i3x} - \varepsilon^{i3x})$$
$$= P^3 - 27x^2P + \varepsilon^P \frac{1}{2i}(\varepsilon^{i3x} - \varepsilon^{-i3x}).$$

But $P^3 - 27x^2P = y^3 - 12xy^2 + 48x^2y - 64x^3 - 27x^2y + 108x^3$;

$$\therefore\ z = y^3 - 12xy^2 + 21x^2y + 44x^3 + \varepsilon^{y-4x}\sin 3x.$$

It must be remembered that this is only one of the many solutions of the equation, for we have assigned two special forms of the entirely arbitrary functions.

Three Partial Equations with an Absolute Term.

E.g. $$(D^2 - DD_1 - 6D_1^2)z = xy.$$

As before, the complete solution consists of the Complementary Function (C.F.) or u, and the Particular Integral (P.I.) or w. The C.F. is obtained by considering the absolute term absent and will contain all the arbitrary functions required; in the above case we have found (p. 105) that $u = f(y + 3x) + \varphi(y - 2x)$.

$$w = \frac{1}{(D - 3D_1)(D + 2D_1)} xy.$$

Taking the last factor first,

$$\frac{1}{D + 2D_1} xy \quad \text{or} \quad \frac{1}{D + 2D_1} \varepsilon^{-2xD_1} \varepsilon^{2xD_1} xy = \varepsilon^{-2xD_1} \frac{1}{D} \varepsilon^{2xD_1} xy;$$

but, as noticed on p. 105, $\varepsilon^{2xD_1} xy = x(y + 2x)$ and $\frac{1}{D}(xy + 2x^2) = \frac{x^2y}{2} + \frac{2x^3}{3}$, for the integration refers only to x.

Now $\varepsilon^{-2xD_1}\left(\frac{x^2y}{2}+\frac{2x^3}{3}\right)$ means that $2x$ must be subtracted from any y in the expression;

$$\therefore \quad \frac{1}{D+2D_1}xy = \frac{x^2(y-2x)}{2}+\frac{2x^3}{3} = \frac{x^2y}{2} - \tfrac{1}{3}x^3.$$

Hence the following practical rule for finding the P.I. of the form $\frac{1}{D+nD_1}f(x, y)$:

1. Add nx to y in the function, forming $f(x, y+nx)$.

2. Integrate the function with regard to x, forming $F(x, y)$.

3. Subtract nx from each y in the last function and $F(x, y-nx)$ is obtained.

The function $\frac{x^2y}{2} - \frac{x^3}{3}$ has now to be subjected to the operator $\frac{1}{D-3D_1}$ or $\varepsilon^{3xD_1}\frac{1}{D}\varepsilon^{-3xD_1}$.

By (1) as $n = -3$, $\frac{x^2(y-3x)}{2} - \frac{x^3}{3}$.

By (2), $\frac{x^3y}{6} - \frac{3x^4}{8} - \frac{x^4}{12} = \frac{x^3y}{6} - \frac{11x^4}{24}$.

By (3), $\frac{x^3(y+3x)}{6} - \frac{11x^4}{24} = \frac{1}{6}\left(x^3y + \frac{x^4}{4}\right)$.

The complete solution of $(D^2 - DD_1 - 6D_1^2)z = xy$ is

$$z = f(y+3x) + \varphi(y-2x) + \tfrac{1}{6}\left(x^3y + \frac{x^4}{4}\right).$$

This is the most general method for finding the P.I., but for special forms of the absolute term there are other more rapid methods, some of which will now be described.

I. Absolute Term (or V) Algebraic; e.g. $x^n y^m$.

$$w = (D^r + aD^{r-1}D_1 + bD^{r-2}D_1^2 + \ldots rD_1^r)^{-1}x^ny^m$$

$$= \frac{1}{D^r}\left(1 + a\frac{D_1}{D} + b\frac{D_1^2}{D^2} + \ldots r\frac{D_1^r}{D^r}\right)^{-1}x^ny^m$$

$$= \frac{1}{D^r}\left[1 - \left(a\frac{D_1}{D} + \ldots r\frac{D_1^r}{D^r}\right) + \left(a\frac{D_1}{D} + \ldots r\frac{D_1^r}{D^r}\right)^2 - \ldots\right]x^ny^m.$$

It will be seen that this method gives the same result as the general

method just given. It is unnecessary to include any higher powers of D_1 than the index of y, so if $m > n$, it is more convenient usually to put $\frac{1}{D_1^r}$ outside the bracket, and to expand in terms of $\frac{D}{D_1}$.

(1) Take the previous example $w = (D^2 - DD_1 - 6D_1^2)^{-1}xy$:

$$w = \frac{1}{D^2}\left[1 - \left(\frac{D_1}{D} + \frac{6D_1^2}{D^2}\right)\right]^{-1} xy = \frac{1}{D^2}\left[1 + \frac{D_1}{D} + \frac{6D_1^2}{D^2}\right]xy,$$

$$w = \frac{1}{D^2}\left(xy + \frac{x^2}{2}\right) = \frac{x^3y}{6} + \frac{x^4}{24}.$$

(2) $w = (D^2 - 2DD_1 + D_1^2)^{-1}xy^3$.

As the index of y is greater than that of x, we write

$$w = \frac{1}{D_1^2}\left[1 - \left(2\frac{D}{D_1} - \frac{D^2}{D_1^2}\right)\right]^{-1} xy^3 = \frac{1}{D_1^2}\left[1 + \left(2\frac{D}{D_1} - \frac{D^2}{D_1^2}\right) + \dots\right]xy^3,$$

$$w = \frac{1}{D_1^2}(xy^3 + \tfrac{1}{2}y^4) = \frac{xy^5}{20} + \frac{y^6}{60}.$$

II. *V* Exponential; e.g. ϵ^{ax+by}.

If $F(D, D_1)z = \varepsilon^{ax+by}$,

$$w = \frac{1}{F(D, D_1)}\varepsilon^{ax+by} \quad \text{or} \quad \frac{1}{F(a, b)}\varepsilon^{ax+by};$$

e.g.
$$(D^2 - DD_1 - 6D_1^2)z = \varepsilon^{ax+by}.$$

$$w = \frac{1}{a^2 - ab - 6b^2}\varepsilon^{ax+by}.$$

If $F(a, b) = 0$, proceed as below when the difficulty occurs:

e.g.
$$(D^2 - DD_1 - 6D_1^2)z = \varepsilon^{3x+y}.$$

$$w = \frac{1}{D - 3D_1}\,\frac{1}{D + 2D_1}\varepsilon^{3x+y} = \frac{1}{D - 3D_1}\,\frac{1}{3 + 2}\varepsilon^{3x+y}$$

$$= \frac{1}{5}\,\frac{1}{D - 3D_1}\varepsilon^{3x+y}$$

$$= \frac{\varepsilon^{3x+y}}{5}\,\frac{1}{D}(1) = \tfrac{1}{5}x\varepsilon^{3x+y}.$$

III. $V = \sin(ax + by)$ or $\cos(ax + by)$.

As $D^2 \sin(ax + by) = -a^2 \sin(ax + by)$ and $D_1{}^2 \sin(ax + by) = -b^2 \sin(ax + by)$, and also $DD_1 \sin(ax + by) = -ab \sin(ax + by)$; and similarly with $\cos(ax + by)$; proceed as below.

If $$w = \frac{1}{D^2 - DD_1 - 6D_1{}^2}\{\sin(2y - 3x) + \cos(y + x)\},$$

$$w = \frac{1}{-9 - 6 + 24}\sin(2y - 3x) + \frac{1}{-1 + 1 + 6}\cos(y + x)$$

$$= \tfrac{1}{9}\sin(2y - 3x) + \tfrac{1}{6}\cos(y + x).$$

If $F(a, b) = 0$, the case when necessary should be treated by the general method described above (p. 108).

E.g. $$(D^2 - DD_1 - 6D_1{}^2)z = \cos(y - 2x).$$

$$w = \frac{1}{D + 2D_1}\,\frac{1}{D - 3D_1}\cos(y - 2x)$$

$$= \frac{1}{D + 2D_1}\,\frac{D}{D - 3D_1}\,\frac{1}{D}\cos(y - 2x).$$

It is necessary to convert $\frac{1}{D - 3D_1}$ into $\frac{D}{D - 3D_1}$ so that it may be easily evaluated without altering the form of the function.

$$w = \frac{1}{D + 2D_1}\,\frac{-2}{-2 - 3}\,\frac{1}{D}\cos(y - 2x)$$

$$= \tfrac{2}{5}\frac{1}{D + 2D_1}\varepsilon^{-2xD_1}\varepsilon^{2xD_1}\frac{1}{-2}\sin(y - 2x)$$

$$= -\tfrac{1}{5}\frac{1}{D + 2D_1}\varepsilon^{-2xD_1}\sin y$$

$$= -\tfrac{1}{5}\varepsilon^{-2xD_1}\sin y\frac{1}{D}(1)$$

$$= -\tfrac{1}{5}\sin(y - 2x)(x) \text{ or } -\frac{x}{5}\sin(y - 2x).$$

The complete solution is

$$z = f(y + 3x) + \varphi(y - 2x) - \frac{x}{5}\sin(y - 2x).$$

It should be noted that in every case in which $F(a, b) = 0$, one of the arbitrary functions may be of the form V; in this case $(y - 2x)$ occurs in the C.F. as $\varphi(y - 2x)$.

FACTORS, SQUARES, SQUARE ROOTS, RECIPROCALS

n	Factors	n^2	$\sqrt{n}$	$\frac{1}{n}$	n	Factors	n^2	$\sqrt{n}$	$\frac{1}{n}$
1		1	1	1	27	3^3	729	5·196152	·0370370
2		4	1·414214	·5	28	$2^2.7$	784	5·291503	·0357143
3		9	1·732051	333333	29		841	5·385165	·0344828
4	2^2	16	2	·25	30	2.3.5	900	5·477226	·0333333
5		25	2·236068	·2					
6	2.3	36	2·44949	·166667	31		961	5·567764	·0322581
7		49	2·645751	·1428571	32	2^5	1024	5·656854	·03125
8	2^3	64	2·828427	·125	33	3.11	1089	5·744563	·0303030
9	3^2	81	3	·111111	34	2.17	1156	5·830952	·0294118
10	2.5	100	3·162278	·1	35	5.7	1225	5·91608	·0285714
					36	$2^2.3^2$	1296	6	·0277778
11		121	3·316625	·090909	37		1369	6·082763	·0270270
12	$2^2.3$	144	3·464102	·083333	38	2.19	1444	6·164414	·0263158
13		169	3·605551	·0769231	39	3.13	1521	6·244998	·0256410
14	2.7	196	3·741657	·0714286	40	$2^3.5$	1600	6·324555	·025
15	3.5	225	3·872983	·0666667					
16	2^4	256	4	·0625	41		1681	6·403124	·0243902
17		289	4·123106	·0588235	42	2.3.7	1764	6·480741	·0238095
18	2.3^2	324	4·242641	·0555556	43		1849	6·557439	·0232558
19		361	4·358899	·0526316	44	$2^2.11$	1936	6·63325	·0227273
20	$2^2.5$	400	4·472136	·05	45	$3^2.5$	2025	6·708204	·0222222
					46	2.23	2116	6·782330	·0217391
21	3.7	441	4·582576	·0476190	47		2209	6·855655	·0212766
22	2.11	484	4·690416	·0454545	48	$2^4.3$	2304	6·928203	·0208333
23		529	4·795832	·0434783	49	7^2	2401	7	·0204082
24	$2^3.3$	576	4·898979	·0416667	50	2.5^2	2500	7·071068	·02
25	5^2	625	5	·04					
26	2.13	676	5·099019	·0384615	51	3.17	2601	7·141428	·0196078

FACTORS, SQUARES, SQUARE ROOTS, RECIPROCALS—*Continued*

n	Factors	n^2	$\sqrt{n}$	$\frac{1}{n}$	n	Factors	n^2	$\sqrt{n}$	$\frac{1}{n}$
52	$2^2.13$	2704	7·211103	·0192308	77	7.11	5929	8·774964	·0129870
53		2809	7·28011	·0188679	78	2.3.13	6084	8·831761	·0128205
54	2.3^3	2916	7·348469	·01851852	79		6241	8·888194	·0126582
55	5.11	3025	7·416198	·0181818	80	$2^4.5$	6400	8·944272	·0125
56	$2^3.7$	3136	7·483315	·0178571	81	3^4	6561	9	·0123457
57	3.19	3249	7·549834	·0175439	82	2.41	6724	9·055385	·0121951
58	2.29	3364	7·615773	·0172414	83		6889	9·110434	·0120482
59		3481	7·681146	·0169492	84	$2^2.3.7$	7056	9·165151	·0119048
60	$2^2.3.5$	3600	7·745967	·0166667	85	5.17	7225	9·219544	·0117647
61		3721	7·81025	·0163934	86	2.43	7396	9·273618	·0116279
62	2.31	3844	7·874008	·0161290	87	3.29	7569	9·327379	·0114943
63	$3^2.7$	3969	7·937254	·0158730	88	$2^3.11$	7744	9·380832	·0113636
64	2^6	4096	8	·015625	89		7921	9·433981	·011236
65	5.13	4225	8·062258	·0153846	90	$2.3^2.5$	8100	9·486833	·0111111
66	2.3.11	4356	8·124038	·0151515	91	7.13	8281	9·539392	·0109890
67		4489	8·185353	·0149254	92	$2^2.23$	8464	9·591663	·0108696
68	$2^2.17$	4624	8·246211	·0147059	93	3.31	8649	9·643651	·0107527
69	3.23	4761	8·306624	·0144928	94	2.47	8836	9·69536	·0106383
70	2.5.7	4900	8·366600	·0142857	95	5.19	9025	9·746794	·0105263
71		5041	8·42615	·0140845	96	$2^5.3$	9216	9·797959	·0104167
72	$2^3.3^2$	5184	8·485281	·0138889	97		9409	9·848858	·0103093
73		5329	8·544004	·0136986	98	2.7^2	9604	9·899495	·0102041
74	2.37	5476	8·602325	·0135135	99	$3^2.11$	9801	9·949874	·0101010
75	3.5^2	5625	8·660254	·0133333	100	$2^2.5^2$	10000	10	·01
76	$2^2.19$	5776	8·717798	·0131579					

If the number of digits be less than the allotted number of decimal places, the real value is somewhat less than that given: **e.g.** the real value of $\sqrt{6}$ is 2·44948974..; in the table is found 2·44949; but the real value of $\sqrt{70}$ is 8·366600265..—greater than that in the table, which is given as 8·366600.

CONVERSION FACTORS

Common and Napierian Logarithms.

Note $\lg n = m \log n$; $\log n = M \lg n$

	$m = \lg 10$	$M = \log e$
1	2·3025,8509	0·4342,9448
2	4·6051,7019	0·8685,8896
3	6·9077,5528	1·3028,8345
4	9·2103,4037	1·7371,7793
5	11·5129,2546	2·1714,7241
6	13·8155,1056	2·6057,6689
7	16·1180,9565	3·0400,6137
8	18·4206,8074	3·4743,5586
9	20·7232,6584	3·9086,5034

Degrees and Radians.

Note π radians (R) are 180°,

$$\text{so } R = \frac{180^\circ}{\pi} = 57^\circ{\cdot}2957795 \text{ or } 57^\circ\ 17'\ 44{\cdot}806''$$

	$\frac{\pi}{180}$		$\frac{\pi}{180 \times 60}$		$\frac{\pi}{180 \times 60 \times 60}$
1°	·0174,5329	1′	·0002,9089	1″	·0000,0485
2°	·0349,0659	2′	·0005,8178	2″	·0000,0970
3°	·0523,5988	3′	·0008,7266	3″	·0000,1454
4°	·0698,1317	4′	·0011,6355	4″	·0000,1939
5°	·0872,6646	5′	·0014,5444	5″	·0000,2424
6°	·1047,1976	6′	·0017,4533	6″	·0000,2909
7°	·1221,7305	7′	·0020,3622	7″	·0000,3394
8°	·1396,2634	8′	·0023,2711	8″	·0000,3879
9°	·1570,7963	9′	·0026,1799	9″	·0000,4363
	Radians		Radians		Radians

INVERSE FUNCTIONS

$$\sin^{-1}n = \sin^{-1}n = \cos^{-1}\sqrt{1-n^2} = \tan^{-1}\frac{n}{\sqrt{1-n^2}} = \cot^{-1}\frac{\sqrt{1-n^2}}{n} = \sec^{-1}\frac{1}{\sqrt{1-n^2}} = \operatorname{cosec}^{-1}\frac{1}{n}$$

$$\cos^{-1}n = \sin^{-1}\sqrt{1-n^2} = \cos^{-1}n = \tan^{-1}\frac{\sqrt{1-n^2}}{n} = \cot^{-1}\frac{n}{\sqrt{1-n^2}} = \sec^{-1}\frac{1}{n} = \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-n^2}}$$

$$\tan^{-1}n = \sin^{-1}\frac{n}{\sqrt{1+n^2}} = \cos^{-1}\frac{1}{\sqrt{1+n^2}} = \tan^{-1}n = \cot^{-1}\frac{1}{n} = \sec\sqrt{1+n^2} = \operatorname{cosec}^{-1}\frac{\sqrt{1+n^2}}{n}$$

$$\cot^{-1}n = \sin^{-1}\frac{1}{\sqrt{1+n^2}} = \cos^{-1}\frac{n}{\sqrt{1+n^2}} = \tan^{-1}\frac{1}{n} = \cot^{-1}n = \sec^{-1}\frac{\sqrt{1+n^2}}{n} = \operatorname{cosec}^{-1}\sqrt{1+n^2}$$

$$\sec^{-1}n = \sin\frac{\sqrt{n^2-1}}{n} = \cos^{-1}\frac{1}{n} = \tan^{-1}\sqrt{n^2-1} = \cot^{-1}\frac{1}{\sqrt{n^2-1}} = \sec^{-1}n = \operatorname{cosec}^{-1}\frac{n}{\sqrt{n^2-1}}$$

$$\operatorname{cosec}^{-1}n = \sin^{-1}\frac{1}{n} = \cos^{-1}\frac{\sqrt{n^2-1}}{n} = \tan^{-1}\frac{1}{\sqrt{n^2-1}} = \cot^{-1}\sqrt{n^2-1} = \sec^{-1}\frac{n}{\sqrt{n^2-1}} = \operatorname{cosec}^{-1}n.$$

The angle whose sine is the number n is called the inverse sine of that *number* n; the expression $\sin^{-1}n$ denotes an angle, not a number. Trigonometrical identities can also be read off from this table. In the column for the given function find the row for the required function and replace n by this function.

E.g. $$\tan\theta = \sqrt{\sec^2\theta - 1};\quad \sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}};\quad \sec\theta = \frac{\sqrt{1+\cot^2\theta}}{\cot\theta}.$$

TRIGONOMETRICAL RATIOS, ETC.

Angle		Sin	Cos	Tan	Cot	Sec	Cosec		
Deg.	Radians								
0°	0	0	1	0	∞	1	∞	1·57080	90°
1°	·01745	·01745	·99985	·01746	57·2900	1·00015	57·2987	1·55334	89°
2°	·03491	·03490	·99939	·03492	28·6363	1·00061	28·6537	1·53589	88°
3°	·05236	·05234	·99863	·05241	19·0811	1·00137	19·1073	1·51844	87°
4°	·06981	·06976	·99756	·06993	14·3007	1·00244	14·3356	1·50098	86°
5°	·08727	·08716	·99619	·08749	11·4301	1·00382	11·4737	1·48353	85°
6°	·10472	·10453	·99452	·10510	9·51436	1·00551	9·56677	1·46608	84°
7°	·12217	·12187	·99255	·12278	8·14435	1·00751	8·20551	1·44862	83°
8°	·13963	·13917	·99027	·14054	7·11537	1·00983	7·18530	1·43117	82°
9°	·15708	·15643	·98769	·15838	6·31375	1·01247	6·39245	1·41372	81°
10°	·17453	·17365	·98481	·17633	5·67128	1·01543	5·75877	1·39626	80°
11°	·19199	·19081	·98163	·19438	5·14455	1·01872	5·24084	1·37881	79°
12°	·20944	·20791	·97815	·21256	4·70463	1·02234	4·80973	1·36136	78°
13°	·22689	·22495	·97437	·23087	4·33148	1·02630	4·44541	1·34390	77°
14°	·24435	·24192	·97030	·24933	4·01078	1·03061	4·13357	1·32645	76°
15°	·26180	·25882	·96593	·26795	3·73205	1·03528	3·86370	1·30900	75°
16°	·27925	·27564	·96126	·28675	3·48741	1·04030	3·62796	1·29154	74°
17°	·29671	·29237	·95630	·30573	3·27085	1·04569	3·42030	1·27409	73°
18°	·31416	·30902	·95106	·32492	3·07768	1·05146	3·23607	1·25664	72°
19°	·33161	·32557	·94552	·34433	2·90421	1·05762	3·07155	1·23918	71°
20°	·34907	·34202	·93969	·36397	2·74748	1·06418	2·92380	1·22173	70°
21°	·36652	·35837	·93358	·38386	2·60509	1·07115	2·79043	1·20428	69°
22°	·38397	·37461	·92718	·40403	2·47509	1·07853	2·66947	1·18682	68°
23°	·40143	·39073	·92050	·42447	2·35585	1·08636	2·55930	1·16937	67°
24°	·41888	·40674	·91355	·44523	2·24604	1·09464	2·45859	1·15192	66°
25°	·43633	·42262	·90631	·46631	2·14451	1·10338	2·36620	1·13446	65°
26°	·45379	·43837	·89879	·48773	2·05030	1·11260	2·28117	1·11701	64°
27°	·47124	·45399	·89101	·50953	1·96261	1·12233	2·20269	1·09956	63°
28°	·48869	·46947	·88295	·53171	1·88073	1·13257	2·13005	1·08210	62°
29°	·50615	·48481	·87462	·55431	1·80405	1·14335	2·06267	1·06465	61°
30°	·52360	·50000	·86603	·57735	1·73205	1·15470	2·00000	1·04720	60°
31°	·54105	·51504	·85717	·60086	1·66428	1·16663	1·94160	1·02974	59°
32°	·55851	·52992	·84805	·62487	1·60033	1·17918	1·88708	1·01229	58°
33°	·57596	·54464	·83867	·64941	1·53987	1·19236	1·83608	·99484	57°
34°	·59341	·55919	·82904	·67451	1·48256	1·20622	1·78829	·97738	56°
35°	·61087	·57358	·81915	·70021	1·42815	1·22077	1·74345	·95993	55°
36°	·62832	·58779	·80902	·72654	1·37638	1·23607	1·70130	·94248	54°
37°	·64577	·60182	·79864	·75355	1·32704	1·25214	1·66164	·92502	53°
38°	·66323	·61566	·78801	·78129	1·27994	1·26902	1·62427	·90757	52°
39°	·68068	·62932	·77715	·80978	1·23490	1·28676	1·58902	·89012	51°
40°	·69813	·64279	·76604	·83910	1·19175	1·30541	1·55572	·87266	50°
41°	·71559	·65606	·75471	·86929	1·15037	1·32501	1·52425	·85521	49°
42°	·73304	·66913	·74314	·90040	1·11061	1·34563	1·49448	·83776	48°
43°	·75049	·68200	·73135	·93252	1·07237	1·36733	1·46628	·82030	47°
44°	·76794	·69466	·71934	·96569	1·03553	1·39016	1·43956	·80285	46°
45°	·78540	·70711	·70711	1·00000	1·00000	1·41421	1·41421	·78540	45°
								Radians	Deg.
		Cos	Sin	Cot	Tan	Cosec	Sec.	Angle	

SOME RATIOS IN SURD-FORM

	Sine	Cosine	Tangent	
9°	$\frac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4}$	$\frac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4}$	$\frac{4-\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	81°
15°	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$2-\sqrt{3}$	75°
18°	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	72°
$22\frac{1}{2}$°	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\sqrt{2}-1$	$67\frac{1}{2}$°
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	60°
36°	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}$	54°
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	45°
	Cosine	Sine	Cotangent	

All ratios of an angle (θ) are the *co*-ratios of its complement $\left(\frac{\pi}{2}-\theta\right)$ and vice versa.

Thus $\sin 9° = \cos 81°$, $\operatorname{cosec} 20° = \sec 70°$, $\tan 60° = \cot 30°$.

As the cosecant, secant, and cotangent are the reciprocals of the sine, cosine, and tangent, all the trigonometrical functions of these angles can be read off from this table:

e.g.
$$\cot 22\tfrac{1}{2}° = \frac{1}{\tan 22\frac{1}{2}°} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1;$$

$$\tan 75° = \frac{1}{\cot 75°} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

General Angle for all Trigonometrical Functions.

If α be an angle in any quadrant,

$$\left.\begin{aligned}\sin\theta &= \sin\alpha\\ \operatorname{cosec}\theta &= \operatorname{cosec}\alpha\end{aligned}\right\}\text{ when } \theta = n\pi + (-1)^n\alpha.$$

$$\left.\begin{aligned}\cos\theta &= \cos\alpha\\ \sec\theta &= \sec\alpha\end{aligned}\right\}\text{ when } \theta = 2n\pi \pm \alpha.$$

$$\left.\begin{aligned}\tan\theta &= \tan\alpha\\ \cot\theta &= \cot\alpha\end{aligned}\right\}\text{ when } \theta = n\pi + \alpha.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$90° + \alpha$	$+\cos\alpha$	$-\sin\alpha$	$-\cot\alpha$
$180° + \alpha$	$-\sin\alpha$	$-\cos\alpha$	$+\tan\alpha$
$270° + \alpha$	$-\cos\alpha$	$+\sin\alpha$	$-\cot\alpha$
$90° - \alpha$	$+\cos\alpha$	$+\sin\alpha$	$+\cot\alpha$
$180° - \alpha$	$+\sin\alpha$	$-\cos\alpha$	$-\tan\alpha$
$270° - \alpha$	$-\cos\alpha$	$-\sin\alpha$	$+\cot\alpha$

DERIVATIVES

$$\frac{dx^n}{dx} = nx^{n-1}.$$

$$\frac{d\lg x}{dx} = \frac{1}{x}.$$

$$\frac{d\log x}{dx} = \frac{1}{x}\log e.$$

$$\frac{da^x}{dx} = a^x \lg a.$$

$$\frac{de^{kx}}{dx} = ke^{kx}.$$

INTEGRALS

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

$$\int \frac{dx}{x} = \lg x.$$

$$\int \frac{dx}{x} = \frac{\log x}{\log e}.$$

$$\int a^x\,dx = \frac{a^x}{\lg a}.$$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k}.$$

The addition of a constant to x makes no difference to the *form* of the result;

e.g. $\dfrac{d(x+a)^n}{dx} = n(x+a)^{n-1}$; $\displaystyle\int \frac{dx}{x+a} = \lg(x+a)$.

If x be multiplied by a constant factor k, the derivative is multiplied by that factor, but the integral is divided by it;

e.g. $\dfrac{d\lg(kx+a)}{dx} = \dfrac{k}{kx+a}$; $\displaystyle\int (kx+a)^n\,dx = \frac{(kx+a)^{n+1}}{k(n+1)}$.

CIRCULAR FUNCTIONS

DERIVATIVES	INTEGRALS
$\dfrac{d\sin nx}{dx} = n\cos nx$.	$\displaystyle\int \cos nx\,dx = \frac{1}{n}\sin nx$.
$\dfrac{d\cos nx}{dx} = -n\sin nx$.	$\displaystyle\int \sin nx\,dx = -\frac{1}{n}\cos nx$.
$\dfrac{d\tan nx}{dx} = n\sec^2 nx$.	$\displaystyle\int \sec^2 nx\,dx = \frac{1}{n}\tan nx$.
$\dfrac{d\,\mathrm{cosec}\,nx}{dx} = -\dfrac{n\cos nx}{\sin^2 nx}$.	$\displaystyle\int \frac{\cos nx\,dx}{\sin^2 nx} = -\frac{1}{n}\mathrm{cosec}\,nx$.
$\dfrac{d\sec nx}{dx} = \dfrac{n\sin nx}{\cos^2 nx}$.	$\displaystyle\int \frac{\sin nx\,dx}{\cos^2 nx} = \frac{1}{n}\sec nx$.
$\dfrac{d\cot nx}{dx} = -n\,\mathrm{cosec}^2 nx$.	$\displaystyle\int \mathrm{cosec}^2 nx\,dx = -\frac{1}{n}\cot nx$.
	$\displaystyle\int \tan nx\,dx = \frac{1}{n}\lg\sec nx$.
	$\displaystyle\int \cot nx\,dx = \frac{1}{n}\lg\sin nx$.
	$\displaystyle\int \mathrm{cosec}\,nx\,dx = \frac{1}{n}\lg\tan\frac{nx}{2}$.
	$\displaystyle\int \sec nx\,dx = \frac{1}{n}\lg\tan\left(\frac{\pi}{4}+\frac{nx}{2}\right)$.

Note that the derivatives of all the co-ratios in this table are negative.

INVERSE CIRCULAR FUNCTIONS

DERIVATIVES	INTEGRALS
$\dfrac{d\sin^{-1}\frac{nx}{a}}{dx} = \dfrac{n}{\sqrt{a^2 - n^2x^2}}.$	$\displaystyle\int \frac{dx}{\sqrt{a^2 - n^2x^2}} = \frac{1}{n}\sin^{-1}\frac{nx}{a},$ or $-\dfrac{1}{n}\cos^{-1}\dfrac{nx}{a}.$
$\dfrac{d\cos^{-1}\frac{nx}{a}}{dx} = \dfrac{-n}{\sqrt{a^2 - n^2x}}.$	
$\dfrac{d\tan^{-1}\frac{nx}{a}}{dx} = \dfrac{na}{a^2 + n^2x^2}.$	$\displaystyle\int \frac{dx}{a^2 + n^2x^2} = \frac{1}{na}\tan^{-1}\frac{nx}{a},$ or $-\dfrac{1}{na}\cot^{-1}\dfrac{nx}{a}.$
$\dfrac{d\cot^{-1}\frac{nx}{a}}{dx} = \dfrac{-na}{a^2 + n^2x^2}.$	
$\dfrac{d\sec^{-1}\frac{nx}{a}}{dx} = \dfrac{a}{x\sqrt{n^2x^2 - a^2}}.$	$\displaystyle\int \frac{dx}{x\sqrt{n^2x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{nx}{a},$ or $-\dfrac{1}{a}\operatorname{cosec}^{-1}\dfrac{nx}{a}.$
$\dfrac{d\operatorname{cosec}^{-1}\frac{nx}{a}}{dx} = \dfrac{-a}{x\sqrt{n^2x^2 - a^2}}.$	

The angle denoted by an inverse circular function is not $> \frac{1}{2}\pi$ and must be measured in radians; the angle is found from the ordinary table in degrees, and it must then be converted into radians (p. 114). If the angle be small, 5° or so, the sine of the angle may be taken as its value in radians; sin 10° is less than 0·51 per cent smaller than the true value of 10° in radians.

All inverse ratios are equal to $\frac{1}{2}\pi$ — their inverse co-ratios; e.g. $\sin^{-1} n = \frac{1}{2}\pi - \cos^{-1} n$; this $\frac{1}{2}\pi$ can be ncluded in the arbitrary constant,

so the double form of the integral is explained. Negative integrands in this table will be expressed either by an inverse co-ratio or by a negative inverse ratio:

e.g. $$-\int \frac{dx}{\sqrt{a^2 - n^2 x^2}} = \frac{1}{n} \cos^{-1} \frac{nx}{a}, \quad \text{or} \quad -\frac{1}{n} \sin^{-1} \frac{nx}{a}.$$

HYPERBOLIC FUNCTIONS

Derivatives

$$\frac{d \sinh mx}{dx} = m \cosh mx.$$

$$\frac{d \cosh mx}{dx} = m \sinh mx.$$

$$\frac{d \tanh mx}{dx} = m \operatorname{sech}^2 mx.$$

$$\frac{d \operatorname{cosech} mx}{dx} = -\frac{m \cosh mx}{\sinh^2 mx}.$$

$$\frac{d \operatorname{sech} mx}{dx} = -\frac{m \sinh mx}{\cosh^2 mx}.$$

$$\frac{d \coth mx}{dx} = -m \operatorname{cosech}^2 mx.$$

Integrals

$$\int \cosh mx \, dx = \frac{1}{m} \sinh mx.$$

$$\int \sinh mx \, dx = \frac{1}{m} \cosh mx.$$

$$\int \operatorname{sech}^2 mx \, dx = \frac{1}{m} \tanh mx.$$

$$\int \frac{\cosh mx}{\sinh^2 mx} dx = -\frac{1}{m} \operatorname{cosech} mx.$$

$$\int \frac{\sinh mx}{\cosh^2 mx} dx = -\frac{1}{m} \operatorname{sech} mx.$$

$$\int \operatorname{cosech}^2 mx \, dx = -\frac{1}{m} \coth mx.$$

$$\int \tanh mx \, dx = \frac{1}{m} \lg \cosh mx.$$

$$\int \operatorname{cosech} mx \, dx = \frac{1}{m} \lg \tanh \frac{mx}{2}.$$

$$\int \operatorname{sech} mx \, dx = \frac{2}{m} \tan^{-1} e^{mx}.$$

$$\int \coth mx \, dx = \frac{1}{m} \lg \sinh mx.$$

INVERSE HYPERBOLIC FUNCTIONS

Derivatives | **Integrals**

$$\frac{d\sinh^{-1}\frac{mx}{a}}{dx} = \frac{m}{\sqrt{m^2x^2+a^2}}. \qquad \int\frac{dx}{\sqrt{m^2x^2+a^2}} = \frac{1}{m}\sinh^{-1}\frac{mx}{a}, \text{ or } \frac{1}{m}\lg\frac{mx+\sqrt{m^2x^2+a^2}}{a}.$$

$$\frac{d\cosh^{-1}\frac{mx}{a}}{dx} = \frac{m}{\sqrt{m^2x^2-a^2}}. \qquad \int\frac{dx}{\sqrt{m^2x^2-a^2}} = \frac{1}{m}\cosh^{-1}\frac{mx}{a}, \text{ or } \frac{1}{m}\lg\frac{mx+\sqrt{m^2x^2-a^2}}{a}.$$

$$\frac{d\tanh^{-1}\frac{mx}{a}}{dx} = \frac{ma}{a^2-m^2x^2}. \qquad \int\frac{dx}{a^2-m^2x^2} = \frac{1}{ma}\tanh^{-1}\frac{mx}{a}, \text{ or } \frac{1}{2ma}\lg\frac{a+mx}{a-mx}.$$

$$\frac{d\,\mathrm{cosech}^{-1}\frac{mx}{a}}{dx} = -\frac{a}{x\sqrt{a^2+m^2x^2}}. \qquad \int\frac{dx}{x\sqrt{a^2+m^2x^2}} = -\frac{1}{a}\mathrm{cosech}^{-1}\frac{mx}{a}, \text{ or } \frac{1}{a}\lg\frac{mx}{a+\sqrt{a^2+m^2x^2}}.$$

$$\frac{d\,\mathrm{sech}^{-1}\frac{mx}{a}}{dx} = -\frac{a}{x\sqrt{a^2-m^2x^2}}. \qquad \int\frac{dx}{x\sqrt{a^2-m^2x^2}} = -\frac{1}{a}\mathrm{sech}^{-1}\frac{mx}{a}, \text{ or } \frac{1}{a}\lg\frac{mx}{a+\sqrt{a^2-m^2x^2}}.$$

$$\frac{d\coth^{-1}\frac{mx}{a}}{dx} = -\frac{ma}{m^2x^2-a^2}. \qquad \int\frac{dx}{m^2x^2-a^2} = -\frac{1}{ma}\coth^{-1}\frac{mx}{a}, \text{ or } \frac{1}{2ma}\lg\frac{mx-a}{mx+a}.$$

SOME COMMON INTEGRALS

$$\int \lg x\,dx = x\lg x - x.$$

$$\int x^n \lg x\,dx = x^{n+1}\left[\frac{\lg x}{n+1} - \frac{1}{(n+1)^2}\right].$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\lg\frac{x-a}{x+a}.$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\lg\frac{a+x}{a-x}.$$

$$\int \frac{dx}{\sqrt{x^2\pm a^2}} = \lg\frac{x+\sqrt{x^2\pm a^2}}{a}.$$

$$\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}\left\{x\sqrt{x^2\pm a^2} \pm a^2\lg\frac{(x+\sqrt{x^2\pm a^2})}{a}\right\}.$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right).$$

$$\int \frac{x\,dx}{a\pm bx} = \frac{1}{b^2}[a\pm bx - a\ \lg(a\pm bx)].$$

$$\int \frac{dx}{a\pm b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a\mp b}{a\pm b}}\tan\frac{x}{2}\right).$$

$$\int \frac{dx}{a\pm b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\left(\tan\frac{x}{2}\pm\frac{b}{a}\right)}{\sqrt{a^2-b^2}}.$$

If $a < b$,

$$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}}\lg\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}};$$

the corresponding result for $\int\frac{dx}{a+b\sin x}$ can be deduced from this by putting x equal to $\frac{\pi}{2}-\theta$.

LOG $\Gamma(p)$

p	0	1	2	3	4	5	6	7	8	9
1·00	0	9750	9500	9251	9003	8756	8509	8263	8017	7773
1·01	$\bar{1}$·997529	7285	7043	6801	6560	6320	6080	5841	5603	5365
1·02	5128	4892	4656	4421	4187	3954	3721	3489	3257	3026
1·03	2796	2567	2338	2110	1883	1656	1430	1205	0981	0757
1·04	0533	0311	0089	9868	9647	9427	9208	8989	8772	8554
1·05	$\bar{1}$·988338	8122	7907	7692	7478	7265	7052	6841	6629	6419
1·06	6209	6000	5791	5583	5376	5169	4963	4758	4553	4349
1·07	4145	3943	3741	3539	3338	3138	2939	2740	2541	2344
1·08	2147	1951	1755	1560	1366	1172	0979	0786	0594	0403
1·09	0212	0022	9833	9644	9456	9269	9082	8896	8710	8525
1·10	$\bar{1}$·978341	8157	7974	7791	7610	7428	7248	7068	6888	6710
1·11	6531	6354	6177	6000	5825	5650	5475	5301	5128	4955
1·12	4783	4612	4441	4271	4101	3932	3764	3596	3429	3262
1·13	3096	2931	2766	2602	2438	2275	2113	1951	1790	1629
1·14	1469	1309	1150	0992	0835	0677	0521	0365	0210	0055
1·15	$\bar{1}$·969901	9747	9594	9442	9290	9139	8988	8838	8688	8539
1·16	8391	8243	8096	7949	7803	7658	7513	7369	7225	7082
1·17	6939	6797	6655	6514	6374	6234	6095	5957	5818	5681
1·18	5544	5408	5272	5137	5002	4868	4734	4601	4469	4337
1·19	4205	4075	3944	3815	3686	3557	3429	3302	3175	3048
1·20	2922	2797	2672	2548	2425	2302	2179	2057	1936	1815
1·21	1695	1575	1456	1337	1219	1101	0984	0867	0751	0636
1·22	0521	0407	0293	0180	0067	9955	9843	9732	9621	9511
1·23	$\bar{1}$·959402	9292	9184	9076	8968	8861	8755	8649	8544	8439
1·24	8335	8231	8128	8025	7923	7821	7720	7620	7520	7420

LOG $\Gamma(p)$—*Continued*

p	0	1	2	3	4	5	6	7	8	9
1·25	$\bar{1}$·957321	7223	7125	7027	6930	6854	6738	6642	6547	6453
1·26	6359	6266	6173	6081	5989	5898	5807	5716	5627	5537
1·27	5449	5360	5273	5185	5099	5013	4927	4842	4757	4673
1·28	4589	4506	4423	4341	4259	4178	4097	4017	3938	3858
1·29	3780	3702	3624	3547	3470	3394	3318	3243	3168	3094
1·30	3020	2947	2874	2802	2730	2659	2588	2518	2448	2379
1·31	2310	2242	2174	2107	2040	1973	1907	1842	1777	1713
1·32	1649	1585	1522	1459	1397	1336	1275	1214	1154	1094
1·33	1035	0977	0918	0861	0803	0747	0690	0634	0579	0524
1·34	0470	0416	0362	0309	0257	0205	0153	0102	0051	0001
1·35	$\bar{1}$·949951	9902	9853	9805	9757	9710	9663	9617	9571	9525
1·36	9480	9435	9391	9348	9304	9262	9219	9178	9136	9095
1·37	9055	9015	8975	8936	8898	8859	8822	8785	8748	8711
1·38	8676	8640	8605	8571	8537	8503	8470	8437	8405	8373
1·39	8342	8311	8280	8250	8221	8192	8163	8135	8107	8080
1·40	8053	8026	8000	7975	7950	7925	7901	7877	7854	7831
1·41	7808	7786	7765	7744	7723	7703	7683	7664	7645	7626
1·42	7608	7590	7573	7556	7540	7524	7509	7494	7479	7465
1·43	7451	7438	7425	7413	7401	7389	7378	7368	7357	7348
1·44	7338	7329	7321	7313	7305	7298	7291	7284	7278	7273
1·45	7268	7263	7259	7255	7251	7248	7246	7244	7242	7241
1·46	7240	7239	7239	7240	7240	7242	7243	7245	7248	7251
1·47	7254	7258	7262	7266	7271	7277	7282	7289	7295	7302
1·48	7310	7318	7326	7335	7344	7353	7363	7373	7384	7395
1·49	7407	7419	7431	7444	7457	7471	7485	7499	7514	7529

LOG $\Gamma(p)$—*Continued*

p	0	1	2	3	4	5	6	7	8	9
1·50	$\bar{1}$·947545	7561	7577	7594	7612	7629	7647	7666	7685	7704
1·51	7724	7744	7764	7785	7806	7828	7850	7873	7896	7919
1·52	7943	7967	7991	8016	8041	8067	8093	8120	8147	8274
1·53	8202	8230	8258	8287	8316	8346	8376	8406	8437	8468
1·54	8500	8532	8564	8597	8630	8664	8698	8732	8767	8802
1·55	8837	8873	8910	8946	8983	9021	9059	9097	9136	9175
1·56	9214	9254	9294	9334	9375	9417	9458	9500	9543	9586
1·57	9629	9673	9717	9761	9806	9851	9896	9942	9989	0035
1·58	$\bar{1}$·950082	0130	0177	0226	0274	0323	0372	0422	0472	0523
1·59	0573	0625	0676	0728	0780	0833	0886	0940	0993	1048
1·60	1102	1157	1212	1268	1324	1380	1437	1494	1552	1610
1·61	1668	1727	1786	1845	1905	1965	2025	2086	2148	2209
1·62	2271	2333	2396	2459	2523	2586	2650	2715	2780	2845
1·63	2911	2977	3043	3110	3177	3244	3312	3380	3449	3518
1·64	3587	3656	3726	3797	3867	3938	4010	4082	4154	4226
1·65	4299	4372	4446	4520	4594	4668	4743	4819	4894	4970
1·66	5047	5124	5201	5278	5356	5434	5513	5592	5671	5750
1·67	5830	5911	5991	6072	6154	6235	6317	6400	6483	6566
1·68	6649	6733	6817	6902	6986	7072	7157	7243	7329	7416
1·69	7503	7590	7678	7766	7854	7943	8032	8121	8211	8301
1·70	8391	8482	8573	8665	8756	8848	8941	9034	9127	9220
1·71	9314	9408	9503	9598	9693	9788	9884	9981	0077	0174
1·72	$\bar{1}$·960271	0369	0467	0565	0664	0763	0862	0961	1061	1162
1·73	1262	1363	1465	1566	1668	1770	1873	1976	2079	2183
1·74	2287	2391	2496	2601	2706	2812	2918	3024	3131	3238
1·75	3345	3453	3561	3669	3778	3887	3996	4106	4216	4326

LOG $\Gamma(p)$—*Continued*

p	0	1	2	3	4	5	6	7	8	9
1·76	$\bar{1}$·964436	4547	4659	4770	4882	4994	5107	5220	5333	5447
1·77	5561	5675	5789	5904	6020	6135	6251	6367	6484	6600
1·78	6718	6835	6953	7071	7190	7308	7427	7547	7667	7787
1·79	7907	8028	8149	8270	8392	8514	8636	8759	8882	9005
1·80	9129	9253	9377	9501	9626	9752	9877	0003	0129	0256
1·81	$\bar{1}$·970382	0510	0637	0765	0893	1021	1150	1279	1408	1538
1·82	1668	1798	1929	2060	2191	2322	2454	2586	2719	2852
1·83	2985	3118	3252	3386	3520	3655	3790	3925	4061	4197
1·84	4333	4470	4607	4744	4881	5019	5157	5296	5434	5573
1·85	5713	5852	5992	6133	6273	6414	6555	6697	6838	6981
1·86	7123	7266	7409	7552	7696	7840	7984	8129	8273	8419
1·87	8564	8710	8856	9002	9149	9296	9443	9591	9739	9887
1·88	$\bar{1}$·980036	0184	0334	0483	0633	0783	0933	1084	1235	1386
1·89	1537	1689	1841	1994	2147	2300	2453	2607	2761	2915
1·90	3069	3224	3379	3535	3691	3847	4003	4160	4316	4474
1·91	4631	4789	4947	5106	5264	5423	5583	5742	5902	6062
1·92	6223	6383	6545	6706	6868	7029	7192	7354	7517	7680
1·93	7844	8007	8171	8336	8500	8665	8830	8996	9161	9328
1·94	9494	9661	9827	9995	0162	0330	0498	0666	0835	1004
1·95	$\bar{1}$·991173	1343	1513	1683	1853	2024	2195	2366	2538	2709
1·96	2882	3054	3227	3400	3573	3746	3920	4094	4269	4444
1·97	4619	4794	4969	5145	5321	5498	5674	5851	6029	6206
1·98	6384	6562	6741	6919	7098	7277	7457	7637	7817	7997
1·99	8178	8359	8540	8722	8903	9085	9268	9450	9633	9817

These figures have been drawn by permission from the seven-figure table published in the Smithsonian Physical Tables